

LINEAR ABADI & PLOTKIN LOGIC

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ABSTRACT. We present a formalization of a version of Abadi and Plotkin’s logic for parametricity for a polymorphic dual intuitionistic/linear type theory with fixed points, and show, following Plotkin’s suggestions, that it can be used to define a wide collection of types, including existential types, inductive types, coinductive types and general recursive types. We show that the recursive types satisfy a universal property called dinaturality, and we develop reasoning principles for the constructed types. In the case of recursive types, the reasoning principle is a mixed induction/coinduction principle, with the curious property that coinduction holds for general relations, but induction only for a limited collection of “admissible” relations. A similar property was observed in Pitts’ 1995 analysis of recursive types in domain theory. In a future paper we will develop a category theoretic notion of models of the logic presented here, and show how the results developed in the logic can be transferred to the models.

INTRODUCTION

In 1983 Reynolds argued that parametric models of the second-order lambda calculus are very useful for modeling data abstraction in programming [Rey83] (see also [Pie02] for a recent textbook description). For real programming, one is of course not just interested in a strongly terminating calculus such as the second-order lambda calculus, but also in a language with full recursion. Thus in *loc. cit.* Reynolds also asked for a parametric *domain-theoretic* model of polymorphism. Informally, what is meant [Rey00] by this is a model of an extension of the polymorphic lambda calculus [Rey74, Gir72], with a polymorphic fixed-point operator $Y: \forall \alpha. (\alpha \rightarrow \alpha) \rightarrow \alpha$ such that

- (1) types are modeled as domains, the sublanguage without polymorphism is modeled in the standard way and $Y\sigma$ is the least fixed-point operator for the domain σ ;

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- (2) the logical relations theorem (also known as the abstraction theorem) is satisfied when the logical relations are admissible, i.e., strict and closed under limits of chains;
- (3) every value in the domain representing some polymorphic type is parametric in the sense that it satisfies the logical relations theorem (even if it is not the interpretation of any expression of that type).

Of course, this informal description leaves room for different formalizations of the problem. Even so, it has proved to be a non-trivial problem. Unpublished work of Plotkin [Plot93] indicates one way to solve the problem model-theoretically by using strict, admissible partial equivalence relations over a domain model of the untyped lambda calculus but, as far as we know, the details of this relationally parametric model have not been worked out in the literature.

From a type theoretical perspective parametric polymorphism is interesting because it allows for encodings of a large collection of types from a small number of constructions. For example adding parametric polymorphism as a reasoning principle to the second-order lambda calculus gives encodings of products, coproducts, existential types and general inductive and coinductive types from just \rightarrow and polymorphism [PA93, Bg05].

This strength of the typing system also complicates matters when adding recursion. Simply adding a polymorphic fixed point combinator to parametric second order lambda calculus would give a type theory with coproducts, products, function spaces and fixed points, a combination known to exist only in the trivial case of all types being isomorphic [HP90]. Inspired by domain theory Plotkin suggested to consider a polymorphic dual intuitionistic/linear lambda calculus and restrict the parametricity principle accordingly to give encodings of coproducts and (co-)inductive types in the linear part of the calculus but not the intuitionistic part. Moreover, the existence of fixed points would provide solutions to general recursive type equations using Freyd's theory of algebraically compact categories [Fre90b, Fre90a, Fre91]. This led Plotkin to argue that such a calculus could serve as a very powerful metalanguage for domain theory.

Thus parametric domain-theoretic models of polymorphic intuitionistic / linear lambda calculus are of importance both from a programming language perspective (for modeling data abstraction) and from a purely domain-theoretic perspective.

Recently, Pitts and coworkers [BPR00] have presented a syntactic approach to Reynolds' challenge, where the notion of domain is essentially taken to be equivalence classes of terms modulo a particular notion of contextual equivalence derived from an operational semantics for a language called Lily, which is essentially polymorphic intuitionistic/linear lambda calculus endowed with an operational semantics.

In parallel with the work presented here, Rosolini and Simpson [RS04] have shown how to construct parametric domain-theoretic models using synthetic domain-theory in intuitionistic set-theory. Moreover, they have shown how to give a computationally adequate denotational semantics of Lily.

This paper presents a formalization of Abadi & Plotkin's logic adapted to the case of Polymorphic Intuitionistic/Linear Lambda calculus with a polymorphic fixed point combinator denoted Y — a language which we shall call PILL_Y . PILL_Y is a simple extension of Barber and Plotkin's dual intuitionistic/linear lambda calculus (DILL) with polymorphism and fixed points. By dual we mean that terms have two contexts of term variables: an intuitionistic and a linear one.

Linear Abadi-Plotkin Logic (LAPL) presented in this paper is a logic for reasoning about parametricity for PILL_Y . As mentioned above, for the logic to be consistent, the

parametricity principle has to be restricted in some way, so that it can be used to prove universal properties in the category of linear terms, but not in the category of intuitionistic terms. To achieve this restriction, LAPL is equipped with a notion of admissible relation, and the parametricity principle is formulated using these relations only. Admissible relations form a subset of the set of definable relations between types, and the prime example of an admissible relation in the logic is the graph of a linear function, whereas the prime example of a relation that is not admissible in general is the graph of an intuitionistic function.

Using the logic, we show how Plotkin's encodings of a large collection of datatypes satisfy the usual universal properties with respect to linear maps in the calculus, up to provability in the logic. In the case of inductive types this means showing that the encodings give initial algebras for certain functors induced by types, for coinductive types we get final coalgebras, and for the general recursive types, the encodings give initial dialgebras for the bifunctors induced by type expressions. These results were sketched by Plotkin in [Plo93], but since the proofs are non-trivial and have never appeared in the literature we include them here. We treat recursive types in full generality, meaning that we treat recursive types with parameters showing that nested recursive types can be modeled.

We also present reasoning principles for the constructed types. Using parametricity we get an induction principle for inductive types holding only for admissible relations. For the coinductive types we get a coinduction principle holding for all relations. These results are extended to recursive types giving a mixed induction/coinduction principle in which the induction part holds for admissible relations only, but the coinduction part holds for all relations. Again these principles are treated in full generality, i.e., also for recursive types with parameters. A similar induction/coinduction principle with the same restrictions was discovered by Pitts [Pit95] for recursive types in domain theory.

The present paper is the first in a series presenting an axiomatization of domain theoretic models of parametricity. In a forthcoming paper (based on [BMP05]) we present a sound and complete notion of parametric models of LAPL called parametric LAPL-structures, and show how to transfer the results proved in LAPL to these. In further papers we will show examples of such parametric LAPL-structures, first treating Plotkin's idea of using admissible pers over reflexive domains, and in further papers we show how Rosolini and Simpson's construction [RS04] can be seen as constructing parametric LAPL-structures and we construct LAPL-structures from Lily syntax in [BMPV06]. Finally in [Møg05b] we show how the parametric completion process of Robinson & Rosolini [RR94] can be adapted to construct parametric LAPL-structures from internal models of PILLY in quasi toposes.

In each of these models the abstract notion of admissible relations in LAPL is interpreted differently. For example, in the per model the notion of admissible relations are certain subsets of the set of equivalence classes of pers, and in the Lily model admissible relations are $\top\top$ -closed sets of terms. The abstract notion of admissible relations presented in this paper is general enough to fit all these different cases.

We remark that one can see our notion of parametric LAPL-structure as a suitable categorical axiomatization of a good category of domains. In Axiomatic Domain Theory much of the earlier work has focused on axiomatizing the adjunction between the category of predomains and continuous functions and the category of predomains and partial continuous functions [Fio96, Page 7] – here we axiomatize the adjunction between the category of domains and strict functions and the category of domains and all continuous functions and extend it with parametric polymorphism, which then suffices to also model recursive types.

Outline. The remainder of this paper consists of two parts. The first part (Section 1) presents the calculus PILL_Y and the logic LAPL for reasoning about parametricity. The second part (Section 2) gives detailed proofs of correctness of encodings of a series of types including inductive, coinductive and recursive types, and gives the reasoning principles for these.

1. LINEAR ABADI-PLOTKIN LOGIC

In this section we define a logic for reasoning about parametricity for Polymorphic Intuitionistic Linear Lambda calculus with fixed points (PILL_Y). The logic is based on Abadi and Plotkin's logic for parametricity [PA93] for the second-order lambda calculus and thus we refer to the logic as Linear Abadi-Plotkin Logic (LAPL).

The logic for parametricity is basically a higher-order logic over PILL_Y . Expressions of the logic are formulas in contexts of variables of PILL_Y and relations among types of PILL_Y . Thus we start by defining PILL_Y .

1.1. PILL_Y . PILL_Y is essentially Barber and Plotkin's DILL [Bar97] extended with polymorphism and a fixed point combinator.

Well-formed type expressions in PILL_Y are expressions of the form:

$$\alpha_1 : \text{Type}, \dots, \alpha_n : \text{Type} \vdash \sigma : \text{Type}$$

where σ is built using the syntax

$$\sigma ::= \alpha \mid I \mid \sigma \otimes \sigma \mid \sigma \multimap \sigma \mid !\sigma \mid \prod \alpha. \sigma.$$

and all the free variables of σ appear on the left hand side of the turnstile. The last construction binds α , so if we have a type

$$\alpha_1 : \text{Type}, \dots, \alpha_n : \text{Type} \vdash \sigma : \text{Type},$$

then we may form the type

$$\alpha_1 : \text{Type}, \dots, \alpha_{i-1} : \text{Type}, \alpha_{i+1} : \text{Type} \dots \alpha_n : \text{Type} \vdash \prod \alpha_i. \sigma : \text{Type}.$$

We use $\sigma, \tau, \omega, \sigma', \tau' \dots$ to range over types. The list of α 's is called the kind context, and is often denoted simply by Ξ or $\vec{\alpha}$. Since there is only one kind the annotation $: \text{Type}$ is often omitted.

The terms of PILL_Y are of the form:

$$\Xi \mid x_1 : \sigma_1, \dots, x_n : \sigma_n; x'_1 : \sigma'_1, \dots, x'_m : \sigma'_m \vdash t : \tau$$

where the σ_i, σ'_i , and τ are well-formed types in the kind context Ξ . The list of x 's is called the intuitionistic type context and is often denoted Γ , and the list of x' 's is called the linear type context, often denoted Δ . No repetition of variable names is allowed in any of the contexts, but permutation akin to having an exchange rule is. Note, that due to the nature of the axioms of the to-be-introduced formation rules, weakening and contraction can be derived for all but the linear context.

The grammar for terms is:

$$\begin{aligned} t ::= & x \mid \star \mid Y \mid \lambda^\circ x : \sigma. t \mid t t \mid t \otimes t \mid !t \mid \Lambda \alpha : \text{Type}. t \mid t(\sigma) \mid \\ & \text{let } x : \sigma \otimes y : \tau \text{ be } t \text{ in } t \mid \text{let } !x : \sigma \text{ be } t \text{ in } t \mid \text{let } \star \text{ be } t \text{ in } t \end{aligned}$$

We use λ° , which bear some graphical resemblance to \multimap , to denote linear function abstraction. And we use s, t, u, \dots to range over terms.

The formation rules are given in Figure 1. A term context $\Xi \mid \Gamma; \Delta$ is considered well-formed if for all types σ appearing in Γ and Δ , the type construction $\Xi \vdash \sigma : \text{Type}$ is well-formed. The linear contexts Δ and Δ' are considered disjoint if the set of variables appearing in Δ is disjoint from the set of variables appearing in Δ' . We use $-$ to denote an empty context. As the types of variables in the let-constructions and function abstractions are often apparent from the context, these will just as often be omitted.

The fixed point combinator Y appears as a term in the language, but could equivalently have been given as an operator on terms as e.g. the `rec` operator in Lily. By having it as a polymorphic term the parametricity principle it satisfies becomes evident.

$$\begin{array}{c}
 \hline
 \Xi \mid \Gamma; - \vdash \star : I \\
 \hline
 \Xi \mid \Gamma; - \vdash Y : \prod \alpha. !(\alpha \multimap \alpha) \multimap \alpha \\
 \hline
 \Xi \mid \Gamma, x : \sigma; - \vdash x : \sigma \\
 \hline
 \Xi \mid \Gamma, x : \sigma \vdash x : \sigma \\
 \hline
 \Xi \mid \Gamma; \Delta \vdash t : \sigma \multimap \tau \quad \Xi \mid \Gamma; \Delta' \vdash u : \sigma \quad \Delta, \Delta' \text{ disjoint} \\
 \hline
 \Xi \mid \Gamma; \Delta, \Delta' \vdash tu : \tau \\
 \hline
 \Xi \mid \Gamma; \Delta, x : \sigma \vdash u : \tau \\
 \hline
 \Xi \mid \Gamma; \Delta \vdash \lambda^\circ x : \sigma. u : \sigma \multimap \tau \\
 \hline
 \Xi \mid \Gamma; \Delta \vdash t : \sigma \quad \Xi \mid \Gamma; \Delta' \vdash s : \tau \quad \Delta, \Delta' \text{ disjoint} \\
 \hline
 \Xi \mid \Gamma; \Delta, \Delta' \vdash t \otimes s : \sigma \otimes \tau \\
 \hline
 \Xi \mid \Gamma; - \vdash t : \sigma \\
 \hline
 \Xi \mid \Gamma; - \vdash !t : \sigma \\
 \hline
 \Xi, \alpha : \text{Type} \mid \Gamma; \Delta \vdash t : \sigma \quad \Xi \mid \Gamma; \Delta \text{ is well-formed} \\
 \hline
 \Xi \mid \Gamma; \Delta \vdash \Lambda \alpha : \text{Type}. t : \prod \alpha : \text{Type}. \sigma \quad \Xi \mid \Gamma; \Delta \vdash t : \prod \alpha : \text{Type}. \sigma \quad \Xi \vdash \tau : \text{Type} \\
 \hline
 \Xi \mid \Gamma; \Delta \vdash t(\tau) : \sigma[\tau/\alpha] \\
 \hline
 \Xi \mid \Gamma; \Delta \vdash s : \sigma \otimes \sigma' \quad \Xi \mid \Gamma; \Delta', x : \sigma, y : \sigma' \vdash t : \tau \quad \Delta, \Delta' \text{ disjoint} \\
 \hline
 \Xi \mid \Gamma; \Delta, \Delta' \vdash \text{let } x : \sigma \otimes y : \sigma' \text{ be } s \text{ in } t : \tau \\
 \hline
 \Xi \mid \Gamma; \Delta \vdash s : !\sigma \quad \Xi \mid \Gamma, x : \sigma; \Delta' \vdash t : \tau \quad \Delta, \Delta' \text{ disjoint} \\
 \hline
 \Xi \mid \Gamma; \Delta, \Delta' \vdash \text{let } !x : !\sigma \text{ be } s \text{ in } t : \tau \\
 \hline
 \Xi \mid \Gamma; \Delta \vdash t : I \quad \Xi \mid \Gamma; \Delta' \vdash s : \sigma \\
 \hline
 \Xi \mid \Gamma; \Delta, \Delta' \vdash \text{let } \star \text{ be } t \text{ in } s : \sigma
 \end{array}$$

Figure 1: Formation rules for terms

Lemma 1.1. *Any term can in a given context be shown to have at most one type, i.e., if the typing judgements $\Xi \mid \Gamma; \Delta \vdash t: \tau$ and $\Xi \mid \Gamma; \Delta \vdash t: \tau'$ are derivable then $\tau = \tau'$.*

Lemma 1.2. *The following three substitution rules are derivable from the formation rules of PILL_Y .*

$$\frac{\Xi \mid \Gamma; \Delta, x: \sigma \vdash t: \tau \quad \Xi \mid \Gamma; \Delta' \vdash u: \sigma}{\Xi \mid \Gamma; \Delta, \Delta' \vdash t[u/x]: \tau} \quad \frac{\Xi \mid \Gamma, x: \sigma; \Delta \vdash t: \tau \quad \Xi \mid \Gamma; - \vdash u: \sigma}{\Xi \mid \Gamma; \Delta \vdash t[u/x]: \tau} \quad \frac{\Xi, \alpha \mid \Gamma; \Delta \vdash t: \tau \quad \Xi \vdash \sigma: \text{Type}}{\Xi \mid \Gamma[\sigma/\alpha]; \Delta[\sigma/\alpha] \vdash t[\sigma/\alpha]: \tau[\sigma/\alpha]}$$

What we have described above is called *pure* PILL_Y . In general we will consider PILL_Y over polymorphic signatures [Jac99, 8.1.1]. Informally, one may think of such a calculus as pure PILL_Y with added type-constants and term-constants. For instance, one may have a constant type for integers or a constant type for lists $\alpha \vdash \text{lists}(\alpha): \text{Type}$. We will be particularly interested in the internal languages of PILL_Y models which in general will be non-pure calculi.

We will also sometimes speak of the calculus PILL . This is PILL_Y without the fixed point combinator Y .

1.1.1. Equality. The *external equality* relation on PILL_Y terms is the least equivalence relation given by the rules in Figure 2. External equality is typed in the sense that if in a given context two terms are externally equal, then they have the same type. The definition makes use of the notion of a *context*, which, loosely speaking, is a term with exactly one hole in it. Formally contexts are defined using the grammar:

$$\begin{aligned} C[-] ::= & - \mid \text{let } \star \text{ be } C[-] \text{ in } t \mid \text{let } \star \text{ be } t \text{ in } C[-] \mid t \otimes C[-] \mid C[-] \otimes t \mid \\ & \text{let } x \otimes y \text{ be } C[-] \text{ in } t \mid \text{let } x \otimes y \text{ be } t \text{ in } C[-] \mid \lambda^\circ x: \sigma. C[-] \mid \\ & C[-] t \mid t C[-] \mid !C[-] \mid \text{let } !x \text{ be } C[-] \text{ in } t \mid \text{let } !x \text{ be } t \text{ in } C[-] \mid \\ & \Lambda\alpha: \text{Type}. C[-] \mid C[-]\sigma \end{aligned}$$

A context $C[-]$ is called a $\Xi \mid \Gamma; \Delta \vdash \sigma \multimap \Xi \mid \Gamma'; \Delta' \vdash \tau$ context if for any well-formed term $\Xi \mid \Gamma; \Delta \vdash t: \sigma$, the term $\Xi \mid \Gamma'; \Delta' \vdash C[t]: \tau$ is well-formed. A context is *linear*, if it does not contain a subcontext of the form $!C[-]$.

We prove a couple of useful lemmas about external equality.

Lemma 1.3. *Suppose $\Xi \mid \Gamma; \Delta \vdash f, g: !\sigma \multimap \tau$ are terms such that*

$$\Xi \mid \Gamma, x: \sigma; \Delta \vdash f(!x) = g(!x).$$

Then $f = g$.

Proof. Using the rules for external equality, we conclude from the assumption that

$$\Xi \mid \Gamma; \Delta, y: !\sigma \vdash \text{let } !x \text{ be } y \text{ in } f(!x) = \text{let } !x \text{ be } y \text{ in } g(!x)$$

and further that

$$\Xi \mid \Gamma; \Delta, y: !\sigma \vdash f(\text{let } !x \text{ be } y \text{ in } !x) = g(\text{let } !x \text{ be } y \text{ in } !x).$$

Thus

$$\Xi \mid \Gamma; \Delta, y: !\sigma \vdash f(y) = g(y),$$

$$\begin{array}{c}
 \frac{}{\Xi \mid \Gamma; \Delta \vdash (\lambda^\circ x: \sigma. t)u = t[u/x]} \beta\text{-term} \\
 \frac{}{\Xi \mid \Gamma; \Delta \vdash (\Lambda\alpha: \text{Type}. t)\sigma = t[\sigma/\alpha]} \beta\text{-type} \\
 \frac{}{\Xi \mid \Gamma; \Delta \vdash \lambda^\circ x: \sigma. (tx) = t} \eta\text{-term} \\
 \frac{}{\Xi \mid \Gamma; \Delta \vdash \Lambda\alpha: \text{Type}. (t\alpha) = t} \eta\text{-type} \\
 \frac{}{\Xi \mid \Gamma; \Delta \vdash \text{let } \star \text{ be } \star \text{ in } t = t} \beta - \star \\
 \frac{}{\Xi \mid \Gamma; \Delta \vdash \text{let } \star \text{ be } t \text{ in } \star = t} \eta - \star \\
 \frac{}{\Xi \mid \Gamma; \Delta \vdash \text{let } x \otimes y \text{ be } s \otimes u \text{ in } t = t[s, u/x, y]} \beta - \otimes \\
 \frac{}{\Xi \mid \Gamma; \Delta \vdash \text{let } x \otimes y \text{ be } t \text{ in } x \otimes y = t} \eta - \otimes \\
 \frac{}{\Xi \mid \Gamma; \Delta \vdash \text{let } !x: \sigma \text{ be } !u \text{ in } t = t[u/x]} \beta - ! \\
 \frac{}{\Xi \mid \Gamma; \Delta \vdash \text{let } !x: \sigma \text{ be } t \text{ in } !x = t} \eta - ! \\
 \frac{\Xi \mid \Gamma; \Delta \vdash t = s: \sigma \quad C[-] \text{ is a } \Xi \mid \Gamma; \Delta \vdash \sigma - \Xi \mid \Gamma'; \Delta' \vdash \tau \text{ context}}{\Xi \mid \Gamma'; \Delta' \vdash C[t] = C[s]} \\
 \frac{}{C[-] \text{ is a linear context}} \\
 \frac{\Xi \mid \Gamma; \Delta \vdash \text{let } \star \text{ be } t \text{ in } C[u] = C[\text{let } \star \text{ be } t \text{ in } u]}{C[-] \text{ is a linear context and does not bind } x, y \text{ or contain them free}} \\
 \frac{\Xi \mid \Gamma; \Delta \vdash \text{let } x \otimes y \text{ be } t \text{ in } C[u] = C[\text{let } x \otimes y \text{ be } t \text{ in } u]}{C[-] \text{ is linear and does not bind } x \text{ or contain it free}} \\
 \frac{\Xi \mid \Gamma; \Delta \vdash \text{let } !x \text{ be } t \text{ in } C[u] = C[\text{let } !x \text{ be } t \text{ in } u]}{\Xi \mid \Gamma; - \vdash f: !\sigma \multimap \sigma} \\
 \frac{}{\Xi \mid \Gamma; - \vdash f !(Y \sigma (!f)) = Y \sigma (!f)}
 \end{array}$$

Figure 2: Rules for external equality

and hence $f = \lambda^\circ y: !\sigma. f(y) = \lambda^\circ y: !\sigma. g(y) = g$. □

1.1.2. Intuitionistic lambda abstraction. We encode ordinary intuitionistic lambda abstraction using the Girard encoding $\sigma \rightarrow \tau = !\sigma \multimap \tau$. The corresponding lambda abstraction is defined as

$$\lambda x: \sigma. t = \lambda^\circ y: !\sigma. \text{let } !x \text{ be } y \text{ in } t$$

where y is a fresh variable. This gives us the rule

$$\frac{\Xi \mid \Gamma, x: \sigma; \Delta \vdash t: \tau}{\Xi \mid \Gamma; \Delta \vdash \lambda x: \sigma. t: \sigma \rightarrow \tau}$$

$$\begin{aligned}
\Xi &::= - \mid \Xi, \alpha \vdash \text{Type} & \Gamma &::= - \mid \Gamma, x : \sigma \\
\Theta &::= - \mid \Theta, R : \text{Rel}(\sigma, \tau) \mid \Theta, S : \text{AdmRel}(\sigma, \tau) \\
\Xi &: \text{Ctx} & \Xi &\vdash \sigma : \text{Type} & \Xi \mid \Gamma; \Delta : \text{Ctx} \\
\Xi \mid \Gamma \mid \Theta &: \text{Ctx} & \Xi \mid \Gamma; \Delta &\vdash t : \sigma & \Xi \mid \Gamma; \Delta \vdash t = u \\
\Xi \mid \Gamma \mid \Theta &\vdash \rho : \text{Rel}(\sigma, \tau) & \Xi \mid \Gamma \mid \Theta &\vdash \rho : \text{AdmRel}(\sigma, \tau) \\
\Xi \mid \Gamma \mid \Theta &\vdash \phi : \text{Prop} & \Xi \mid \Gamma \mid \Theta &\mid \phi_1, \dots, \phi_n \vdash \psi
\end{aligned}$$

Figure 3: Types of judgments and grammar for LAPL contexts

For evaluation we have the rule

$$\frac{\Xi \mid \Gamma; - \vdash t : \sigma \quad \Xi \mid \Gamma; \Delta \vdash f : \sigma \rightarrow \tau}{\Xi \mid \Gamma; \Delta \vdash f !t : \tau}$$

and the equality rules give

$$(\lambda x : \sigma. t) !s = t[s/x].$$

Note that using this notation the constant Y can obtain the more familiar looking type

$$Y : \Pi \alpha. (\alpha \rightarrow \alpha) \rightarrow \alpha.$$

This notation also explains the occurrences of the $!$'s in the last rule of Figure 2.

1.2. The logic. As mentioned, expressions of LAPL live in contexts of variables of PILL_Y and relations among types of PILL_Y . The contexts look like this:

$$\Xi \mid \Gamma \mid R_1 : \text{Rel}(\tau_1, \tau'_1), \dots, R_n : \text{Rel}(\tau_n, \tau'_n), S_1 : \text{AdmRel}(\omega_1, \omega'_1), \dots, S_m : \text{AdmRel}(\omega_m, \omega'_m)$$

where $\Xi \mid \Gamma; -$ is a context of PILL_Y and the $\tau_i, \tau'_i, \omega_i, \omega'_i$ are well-formed types in context Ξ , for all i . The list of R 's and S 's is called the relational context and is often denoted Θ . As for the other contexts we do not allow repetition, but do allow permutation of variables.

The concept of admissible relations is taken from domain theory. Intuitively admissible relations relate \perp to \perp and are chain complete.

It is important to note that there is no linear component Δ in the contexts — the point is that the logic only allows for *intuitionistic* (no linearity) reasoning about terms of PILL_Y , whereas PILL_Y terms can behave linearly. This simplification of the logic has been chosen since all parametricity arguments in our knowledge involve purely intuitionistic reasoning.

Propositions in the logic are given by the syntax:

$$\begin{aligned}
\phi &::= (t =_\sigma u) \mid \rho(t, u) \mid \phi \supset \psi \mid \perp \mid \top \mid \phi \wedge \psi \mid \phi \vee \psi \mid \forall \alpha : \text{Type}. \phi \mid \\
&\quad \forall x : \sigma. \phi \mid \forall R : \text{Rel}(\sigma, \tau). \phi \mid \forall S : \text{AdmRel}(\sigma, \tau). \phi \mid \\
&\quad \exists \alpha : \text{Type}. \phi \mid \exists x : \sigma. \phi \mid \exists R : \text{Rel}(\sigma, \tau). \phi \mid \exists S : \text{AdmRel}(\sigma, \tau). \phi
\end{aligned}$$

where ρ is a definable relation (to be defined below). The judgments of the logic are presented in Figure 3. In the following we give formation rules for the above.

Remark 1.4. Our Linear Abadi & Plotkin logic is designed for reasoning about binary relational parametricity. For reasoning about other arities of parametricity, one can easily replace binary relations in the logic by relations of other arities. In the case of unary parametricity, for example, one would then have an interpretation of types as predicates. See also [Tak98, Wad04]

We first have the formation rule for internal equality:

$$\frac{\Xi \mid \Gamma; - \vdash t: \sigma \quad \Xi \mid \Gamma; - \vdash u: \sigma}{\Xi \mid \Gamma \mid \Theta \vdash t =_{\sigma} u: \mathbf{Prop}}$$

Notice here the notational difference between $t = u$ and $t =_{\sigma} u$. The former denotes *external* equality and the latter is a proposition in the logic. The rules for \supset , \vee and \wedge are the usual ones, where \supset denotes implication. \top , \perp are propositions in any context. We use \bowtie for biimplication.

We have the following formation rules for universal quantification:

$$\frac{\Xi \mid \Gamma, x: \sigma \mid \Theta \vdash \phi: \mathbf{Prop}}{\Xi \mid \Gamma \mid \Theta \vdash \forall x: \sigma. \phi: \mathbf{Prop}}$$

$$\frac{\Xi \mid \Gamma \mid \Theta, R: \mathbf{Rel}(\sigma, \tau) \vdash \phi: \mathbf{Prop}}{\Xi \mid \Gamma \mid \Theta \vdash \forall R: \mathbf{Rel}(\sigma, \tau). \phi: \mathbf{Prop}}$$

$$\frac{\Xi \mid \Gamma \mid \Theta, S: \mathbf{AdmRel}(\sigma, \tau) \vdash \phi: \mathbf{Prop}}{\Xi \mid \Gamma \mid \Theta \vdash \forall S: \mathbf{AdmRel}(\sigma, \tau). \phi: \mathbf{Prop}}$$

$$\frac{\Xi, \alpha \mid \Gamma \mid \Theta \vdash \phi: \mathbf{Prop}}{\Xi \mid \Gamma \mid \Theta \vdash \forall \alpha: \mathbf{Type}. \phi: \mathbf{Prop}} \quad \Xi \mid \Gamma \mid \Theta \text{ is well-formed}$$

The side condition $\Xi \mid \Gamma \mid \Theta$ is well-formed means that all the types of variables in Γ and of relation variables in Θ are well-formed in Ξ (i.e., all the free type variables of the types occur in Ξ).

There are similar formation rules for the existential quantifier.

Before we give the formation rule for $\rho(t, u)$, we discuss definable relations.

1.2.1. *Definable relations.* Definable relations are given by the grammar:

$$\rho ::= R \mid (x: \sigma, y: \tau). \phi \mid \sigma[R]$$

Definable relations always have a domain and a codomain, just as terms always have types. The basic formation rules for definable relations are:

$$\frac{}{\Xi \mid \Gamma \mid \Theta, R: \mathbf{Rel}(\sigma, \tau) \vdash R: \mathbf{Rel}(\sigma, \tau)}$$

$$\frac{\Xi \mid \Gamma, x: \sigma, y: \tau \mid \Theta \vdash \phi: \mathbf{Prop}}{\Xi \mid \Gamma \mid \Theta \vdash (x: \sigma, y: \tau). \phi: \mathbf{Rel}(\sigma, \tau)}$$

$$\frac{\Xi \mid \Gamma \mid \Theta \vdash \rho: \mathbf{AdmRel}(\sigma, \tau)}{\Xi \mid \Gamma \mid \Theta \vdash \rho: \mathbf{Rel}(\sigma, \tau)}$$

Notice that in the second rule we can only abstract *intuitionistic* variables to obtain definable relations. In the last rule, $\rho: \mathbf{AdmRel}(\sigma, \tau)$ is an admissible relation, a concept to be discussed below. The rule says that the admissible relations constitute a subset of the definable relations. The last construction of the grammar refers to the relational interpretation of types and will be discussed in Section 1.2.3.

An example of a definable relation is the graph relation of a function:

$$\langle f \rangle = (x: \sigma, y: \tau). fx =_{\tau} y,$$

for $f: \sigma \multimap \tau$. The equality relation eq_{σ} is defined as the graph of the identity map.

If $\rho: \text{Rel}(\sigma, \tau)$ is a definable relation, and we are given terms of the right types, then we may form the proposition stating that the two terms are related by the definable relation:

$$\frac{\Xi \mid \Gamma \mid \Theta \vdash \rho: \text{Rel}(\sigma, \tau) \quad \Xi \mid \Gamma; - \vdash t: \sigma, s: \tau}{\Xi \mid \Gamma \mid \Theta \vdash \rho(t, s): \text{Prop}} \quad (1.1)$$

We shall also write $t\rho s$ for $\rho(t, s)$.

Relations can be reindexed along PILL_Y maps as in the following derivable rule

$$\frac{\Xi \mid \Gamma \mid \Theta \vdash \rho: \text{Rel}(\sigma, \tau) \quad \Xi \mid \Gamma; - \vdash f: \sigma' \multimap \sigma, g: \tau' \multimap \tau}{\Xi \mid \Gamma \mid \Theta \vdash (x: \sigma', y: \tau'). \rho(f x, g y): \text{Rel}(\sigma', \tau')}$$

where x, y are fresh variables. We shall use the shorthand notation $(f, g)^* \rho$ for

$$(x: \sigma', y: \tau'). \rho(f x, g y).$$

1.2.2. Constructions on definable relations. In this subsection we present some constructions on definable relations - one for each type constructor of PILL_Y . These will be used to give a relational interpretation of the types of PILL_Y .

If $\rho: \text{Rel}(\sigma, \tau)$ and $\rho': \text{Rel}(\sigma', \tau')$ define

$$\rho \multimap \rho' = (f: \sigma \multimap \sigma', g: \tau \multimap \tau'). \forall x: \sigma. \forall y: \tau. \rho(x, y) \supset \rho'(f x, g y).$$

for fresh variables x, y, f, g . Then the rule

$$\frac{\Xi \mid \Gamma \mid \Theta \vdash \rho: \text{Rel}(\sigma, \tau), \rho': \text{Rel}(\sigma', \tau')}{\Xi \mid \Gamma \mid \Theta \vdash (\rho \multimap \rho'): \text{Rel}((\sigma \multimap \sigma'), (\tau \multimap \tau'))}$$

is derivable.

If

$$\Xi, \alpha, \beta \mid \Gamma \mid \Theta, R: \text{AdmRel}(\alpha, \beta) \vdash \rho: \text{Rel}(\sigma, \tau)$$

is well-formed and $\Xi \mid \Gamma \mid \Theta$ is well-formed, $\Xi, \alpha \vdash \sigma: \text{Type}$, and $\Xi, \beta \vdash \tau: \text{Type}$ we may define the relation

$$\Xi \mid \Gamma \mid \Theta \vdash \forall(\alpha, \beta, R: \text{AdmRel}(\alpha, \beta)). \rho: \text{Rel}((\prod \alpha: \text{Type}. \sigma), (\prod \beta: \text{Type}. \tau))$$

as

$$\begin{aligned} & \forall(\alpha, \beta, R: \text{AdmRel}(\alpha, \beta)). \rho = \\ & (t: \prod \alpha: \text{Type}. \sigma, u: \prod \beta: \text{Type}. \tau). \forall \alpha, \beta: \text{Type}. \forall R: \text{AdmRel}(\alpha, \beta). \rho(t\alpha, u\beta). \end{aligned}$$

In Section 2 we will show how to encode the type constructors $\otimes, !, I$ using \multimap, \rightarrow and polymorphism as in Figure 5 below. At this point we have not discussed parametricity and so can not use the encodings, but we will still use these for the definitions of the constructions on relations corresponding to \otimes, I and $!$. The relational interpretations of $\otimes, I, !$ are due to Alex Simpson, who also uses this relational interpretation of $!$ in a more general context in [Sim06].

First we define the tensor product of ρ and ρ'

$$\rho \otimes \rho': \text{Rel}((\sigma \otimes \sigma'), (\tau \otimes \tau')),$$

for $\rho: \text{Rel}(\sigma, \tau)$ and $\rho': \text{Rel}(\sigma', \tau')$. We first introduce the map

$$f_{\sigma, \sigma'}: \sigma \otimes \sigma' \multimap \prod \alpha. (\sigma \multimap \sigma' \multimap \alpha) \multimap \alpha$$

defined as

$$f_{\sigma, \sigma'} x = \text{let } x' \otimes x'' : \sigma \otimes \sigma' \text{ be } x \text{ in } \Lambda \alpha. \lambda^\circ h : \sigma \multimap \sigma' \multimap \alpha. h \ x' \ x''.$$

Then we define

$$\rho \otimes \rho' = (f_{\sigma, \sigma'}, f_{\tau, \tau'})^* (\forall (\alpha, \beta, R : \text{AdmRel}(\alpha, \beta)). (\rho \multimap \rho' \multimap R) \multimap R),$$

or, if we write it out,

$$\begin{aligned} \rho \otimes \rho' = & (x : \sigma \otimes \sigma', y : \tau \otimes \tau'). \forall \alpha, \beta, R : \text{AdmRel}(\alpha, \beta). \\ & \forall t : \sigma \multimap \tau \multimap \alpha, t' : \sigma' \multimap \tau' \multimap \beta. (\rho \multimap \rho' \multimap R)(t, t') \supset \\ & R(\text{let } x' \otimes x'' \text{ be } x \text{ in } t \ x' \ x'', \text{let } y' \otimes y'' \text{ be } y \text{ in } t' \ y' \ y''). \end{aligned}$$

As a derivable rule we get

$$\frac{\Xi \mid \Gamma \mid \Theta \vdash \rho : \text{Rel}(\sigma, \tau), \rho' : \text{Rel}(\sigma', \tau')}{\Xi \mid \Gamma \mid \Theta \vdash (\rho \otimes \rho') : \text{Rel}((\sigma \otimes \sigma'), (\tau \otimes \tau'))}$$

Following the same strategy, we define a relation $I_{\text{Rel}} : \text{Rel}(I, I)$ using the map

$$f : I \multimap \prod \alpha. \alpha \multimap \alpha$$

defined as $\lambda^\circ x : I. \text{let } \star \text{ be } x \text{ in } id$, where $id = \Lambda \alpha. \lambda^\circ x : \alpha. x$ and define

$$I_{\text{Rel}} = (f, f)^* (\forall (\alpha, \beta, R : \text{AdmRel}(\alpha, \beta)). R \multimap R),$$

which, if we write it out, is

$$\begin{aligned} & (x : I, y : I). \forall (\alpha, \beta, R : \text{AdmRel}(\alpha, \beta)). \forall z : \alpha, w : \beta. \\ & z R w \supset (\text{let } \star \text{ be } x \text{ in } z) R (\text{let } \star \text{ be } y \text{ in } w). \end{aligned}$$

The relation I_{Rel} types in any context, i.e., $\Xi \mid \Gamma \mid \Theta \vdash I_{\text{Rel}} : \text{Rel}(I, I)$ is derivable for any well-formed context $\Xi \mid \Gamma \mid \Theta$.

The encoding of $!$ in Figure 5 uses \rightarrow , which was defined above as $\sigma \rightarrow \tau = !\sigma \multimap \tau$, but since \rightarrow has a natural relational interpretation, we will still use this to define the relational interpretation of $!$.

For $\rho : \text{Rel}(\sigma, \tau)$ and $\rho' : \text{Rel}(\sigma', \tau')$ we define

$$\rho \rightarrow \rho' = (f : \sigma \rightarrow \sigma', g : \tau \rightarrow \tau'). \forall x : \sigma, y : \tau. \rho(x, y) \supset \rho'(f(!x), g(!y))$$

Now, define for any type σ the map $f_\sigma : !\sigma \multimap \prod \alpha. (\sigma \rightarrow \alpha) \multimap \alpha$ as

$$\lambda^\circ x : !\sigma. \Lambda \alpha. \lambda^\circ g : \sigma \rightarrow \alpha. g(x).$$

The relation $!\rho : \text{Rel}(!\sigma, !\tau)$ is defined as

$$(f_\sigma, f_\tau)^* \forall (\alpha, \beta, R : \text{AdmRel}(\alpha, \beta)). (\rho \rightarrow R) \multimap R.$$

The derivable typing rule is

$$\frac{\Xi \mid \Gamma \mid \Theta \vdash \rho : \text{Rel}(\sigma, \tau)}{\Xi \mid \Gamma \mid \Theta \vdash !\rho : \text{Rel}(!\sigma, !\tau)}$$

Remark 1.5. In [BMP05] we show how the constructions on relations presented in this section gives rise to a PILLY-model of admissible relations. In other words \otimes, \multimap defines a symmetric monoidal structure on relations, $!$ extends this to a linear structure, and $\forall (\alpha, \beta, R : \text{AdmRel}(\alpha, \beta)).$ defines a polymorphic product.

Remark 1.6. The definitions of $\rho \otimes \rho'$ and $!\rho$ involve an implicit admissible closure operator discussed in Section 1.2.5 below. This operator helps secure that the collection of admissible relations is closed under the constructions above (see Proposition 1.7 below).

1.2.3. *Admissible relations.* As mentioned in the introduction, for the theory of parametricity to be consistent in a type theory with recursion the parametricity principle must be weakened. For this purpose we introduce a notion of admissible relations axiomatized in Figure 4. In these rules $\rho \equiv \rho'$ is a shorthand for $\forall x, y. \rho(x, y) \supset \rho'(x, y)$.

$$\begin{array}{c}
\frac{}{\Xi \mid \Gamma \mid \Theta, R: \text{AdmRel}(\sigma, \tau) \vdash R: \text{AdmRel}(\sigma, \tau)} \\
\frac{\Xi \mid \Gamma \mid \Theta \vdash \rho: \text{AdmRel}(\sigma, \tau) \quad \Xi \mid \Gamma; - \vdash t: \sigma' \multimap \sigma, u: \tau' \multimap \tau \quad x, y \notin \Gamma}{\Xi \mid \Gamma \mid \Theta \vdash (x: \sigma', y: \tau'). \rho(t \ x, u \ y): \text{AdmRel}(\sigma', \tau')} \\
\frac{\Xi \mid \Gamma \mid \Theta \vdash \rho, \rho': \text{AdmRel}(\sigma, \tau) \quad x, y \notin \Gamma}{\Xi \mid \Gamma \mid \Theta \vdash (x: \sigma, y: \tau). \rho(x, y) \wedge \rho'(x, y): \text{AdmRel}(\sigma, \tau)} \\
\frac{\Xi \mid \Gamma \mid \Theta \vdash \rho: \text{AdmRel}(\sigma, \tau) \quad x, y \notin \Gamma}{\Xi \mid \Gamma \mid \Theta \vdash (x: \tau, y: \sigma). \rho(y, x): \text{AdmRel}(\tau, \sigma)} \\
\frac{\Xi \mid \Gamma \mid \Theta \vdash (x: \sigma, y: \tau). \top: \text{AdmRel}(\sigma, \tau)}{\Xi \mid \Gamma \mid \Theta \vdash \rho: \text{AdmRel}(\sigma, \tau) \quad \Xi \mid \Gamma \mid \Theta \vdash \phi: \text{Prop} \quad x, y \notin \Gamma} \\
\frac{\Xi \mid \Gamma \mid \Theta \vdash (x: \sigma, y: \tau). \phi \supset \rho(x, y): \text{AdmRel}(\sigma, \tau)}{\Xi, \alpha \mid \Gamma \mid \Theta \vdash \rho: \text{AdmRel}(\sigma, \tau) \quad \Xi \mid \Gamma \mid \Theta \quad \Xi \vdash \sigma: \text{Type} \quad \Xi \vdash \tau: \text{Type} \quad x, y \notin \Gamma} \\
\frac{\Xi \mid \Gamma \mid \Theta \vdash (x: \sigma, y: \tau). \forall \alpha: \text{Type}. \rho(x, y): \text{AdmRel}(\sigma, \tau)}{\Xi \mid \Gamma, z: \omega \mid \Theta \vdash \rho: \text{AdmRel}(\sigma, \tau) \quad x, y \notin \Gamma} \\
\frac{\Xi \mid \Gamma \mid \Theta \vdash (x: \sigma, y: \tau). \forall z: \omega. \rho(x, y): \text{AdmRel}(\sigma, \tau)}{\Xi \mid \Gamma \mid \Theta, R: \text{AdmRel}(\omega, \omega') \vdash \rho: \text{AdmRel}(\sigma, \tau) \quad x, y \notin \Gamma} \\
\frac{\Xi \mid \Gamma \mid \Theta \vdash (x: \sigma, y: \tau). \forall R: \text{AdmRel}(\omega, \omega'). \rho(x, y): \text{AdmRel}(\sigma, \tau)}{\Xi \mid \Gamma \mid \Theta, R: \text{Rel}(\omega, \omega') \vdash \rho: \text{AdmRel}(\sigma, \tau) \quad x, y \notin \Gamma} \\
\frac{\Xi \mid \Gamma \mid \Theta \vdash (x: \sigma, y: \tau). \forall R: \text{Rel}(\omega, \omega'). \rho(x, y): \text{AdmRel}(\sigma, \tau)}{\Xi \mid \Gamma \mid \Theta \vdash \rho: \text{AdmRel}(\sigma, \tau), \rho': \text{Rel}(\sigma, \tau) \quad \Xi \mid \Gamma \mid \Theta \mid \top \vdash \rho \equiv \rho'} \\
\frac{\Xi \mid \Gamma \mid \Theta \vdash \rho': \text{AdmRel}(\sigma, \tau)}{\alpha_1, \dots, \alpha_n \vdash \sigma(\vec{\alpha}): \text{Type} \quad \Xi \mid \Gamma \mid \Theta \vdash \rho_1: \text{AdmRel}(\tau_1, \tau'_1), \dots, \rho_n: \text{AdmRel}(\tau_n, \tau'_n)} \\
\frac{}{\Xi \mid \Gamma \mid \Theta \vdash \sigma[\vec{\rho}]: \text{AdmRel}(\sigma(\vec{\tau}), \sigma(\vec{\tau}'))}
\end{array}$$

Figure 4: Rules for admissible relations

A few comments is needed for the last of the rules in Figure 4. First observe that $\sigma[\vec{\rho}]$ is a syntactic construction and is not obtained by substitution as in [PA93]. Still the notation $\sigma[\rho_1/\alpha_1, \dots, \rho_n/\alpha_n]$ might be more complete, but this quickly becomes overly verbose. In [PA93] $\sigma[\vec{\rho}]$ is to some extent defined inductively on the structure of σ , but in our case that is not enough, since we will need to form $\sigma[\vec{\rho}]$ for type constants (when using the internal language of a model of LAPL). The inductive definition over the type structure is in stead reflected in axioms 1.22 to 1.27.

We call $\sigma[\bar{\rho}]$ the *relational interpretation of the type* σ .

Proposition 1.7. *The class of admissible relations contains all graphs and is closed under the constructions of Section 1.2.2, in fact the following more general rules hold*

$$\begin{array}{c}
 \frac{\Xi \mid \Gamma; - \vdash f: \sigma \multimap \tau}{\Xi \mid \Gamma \mid \Theta \vdash \langle f \rangle: \text{AdmRel}(\sigma, \tau)} \\
 \frac{\Xi \mid \Gamma \mid \Theta \vdash \rho: \text{Rel}(\sigma, \tau), \rho': \text{AdmRel}(\sigma', \tau')}{\Xi \mid \Gamma \mid \Theta \vdash (\rho \multimap \rho'): \text{AdmRel}((\sigma \multimap \sigma'), (\tau \multimap \tau'))} \\
 \frac{\Xi \mid \Gamma \mid \Theta \vdash \rho: \text{Rel}(\sigma, \tau), \rho': \text{AdmRel}(\sigma', \tau')}{\Xi \mid \Gamma \mid \Theta \vdash (\rho \rightarrow \rho'): \text{AdmRel}((\sigma \rightarrow \sigma'), (\tau \rightarrow \tau'))} \\
 \frac{\Xi \mid \Gamma \mid \Theta \vdash \rho: \text{Rel}(\sigma, \tau), \rho': \text{Rel}(\sigma', \tau')}{\Xi \mid \Gamma \mid \Theta \vdash (\rho \otimes \rho'): \text{AdmRel}((\sigma \otimes \sigma'), (\tau \otimes \tau'))} \\
 \frac{\Xi \mid \Gamma \mid \Theta \vdash \rho: \text{Rel}(\sigma, \tau)}{\Xi \mid \Gamma \mid \Theta \vdash !\rho: \text{AdmRel}(!\sigma, !\tau)} \\
 \frac{\Xi \mid \Gamma \mid \Theta \vdash I_{\text{Rel}}: \text{AdmRel}(I, I)}{\Xi, \alpha, \beta \mid \Gamma \mid \Theta, R: \text{AdmRel}(\alpha, \beta) \vdash \rho: \text{AdmRel}(\sigma, \tau) \quad \Xi, \alpha \vdash \sigma: \text{Type} \quad \Xi, \beta \vdash \tau: \text{Type}} \\
 \Xi \mid \Gamma \mid \Theta \vdash \forall(\alpha, \beta, R: \text{AdmRel}(\alpha, \beta)). \rho: \text{AdmRel}((\prod \alpha: \text{Type}. \sigma), (\prod \beta: \text{Type}. \tau))
 \end{array}$$

where the last rule has the side condition that $\Xi \mid \Gamma \mid \Theta$ must be well-formed.

Proof. Graph relations are admissible since equality relations are and admissible relations are closed under reindexing. For the constructions of Section 1.2.2, we just give the proof of \multimap .

We must prove that for $\rho: \text{Rel}(\sigma, \tau), \rho': \text{AdmRel}(\sigma', \tau')$ relations in the same context $\rho \multimap \rho'$ is admissible. Consider first the relation

$$(f: \sigma \multimap \sigma', g: \tau \multimap \tau'). \rho'(f x, g y)$$

in the context where we have added fresh variables $x: \sigma, y: \tau$ to the contexts of ρ, ρ' . This relation is a reindexing of ρ' along the evaluation maps, which are linear, and so the relation is admissible. Since f, g do not occur freely in ρ , also

$$(f: \sigma \multimap \sigma', g: \tau \multimap \tau'). \rho(x, y) \supset \rho'(f x, g y)$$

is admissible, and so since admissible relations are closed under universal quantification, $\rho \multimap \rho'$ is admissible. \square

1.2.4. *Axioms and Rules.* The last judgment in Figure 3 has not yet been mentioned. It says that in the given context, the formulas ϕ_1, \dots, ϕ_n collectively imply ψ . We will often write Φ for ϕ_1, \dots, ϕ_n .

Having specified the language of LAPL, it is time to specify the axioms and inference rules. We have all the usual axioms and rules of predicate logic plus the axioms and rules specified below.

Rules for substitution:

$$\begin{aligned}
\textbf{Rule 1.8.} \quad & \frac{\Xi \mid \Gamma, x: \sigma \mid \Theta \mid \top \vdash \phi \quad \Xi \mid \Gamma \vdash t: \sigma}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash \phi[t/x]} \\
\textbf{Rule 1.9.} \quad & \frac{\Xi \mid \Gamma \mid \Theta, R: \text{Rel}(\sigma, \tau) \mid \top \vdash \phi \quad \Xi \mid \Gamma \mid \Theta \vdash \rho: \text{Rel}(\sigma, \tau)}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash \phi[\rho/R]} \\
\textbf{Rule 1.10.} \quad & \frac{\Xi \mid \Gamma \mid \Theta, S: \text{AdmRel}(\sigma, \tau) \mid \top \vdash \phi \quad \Xi \mid \Gamma \mid \Theta \vdash \rho: \text{AdmRel}(\sigma, \tau)}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash \phi[\rho/S]} \\
\textbf{Rule 1.11.} \quad & \frac{\Xi, \alpha \mid \Gamma \mid \Theta \mid \top \vdash \phi \quad \Xi \vdash \sigma: \text{Type}}{\Xi \mid \Gamma[\sigma/\alpha] \mid \Theta[\sigma/\alpha] \mid \top \vdash \phi[\sigma/\alpha]}
\end{aligned}$$

The *substitution* axiom:

$$\textbf{Axiom 1.12.} \quad \forall \alpha, \beta: \text{Type}. \forall x, x': \alpha. \forall y, y': \beta. \forall R: \text{Rel}(\alpha, \beta). R(x, y) \wedge x =_\alpha x' \wedge y =_\beta y' \supset R(x', y')$$

Rules for \forall -quantification:

$$\begin{aligned}
\textbf{Rule 1.13.} \quad & \frac{\Xi, \alpha \mid \Gamma \mid \Theta \mid \Phi \vdash \psi}{\Xi \mid \Gamma \mid \Theta \mid \Phi \vdash \forall \alpha: \text{Type}. \psi} \Xi \mid \Gamma \mid \Theta \vdash \Phi \\
\textbf{Rule 1.14.} \quad & \frac{\Xi \mid \Gamma, x: \sigma \mid \Theta \mid \Phi \vdash \psi}{\Xi \mid \Gamma \mid \Theta \mid \Phi \vdash \forall x: \sigma. \psi} \Xi \mid \Gamma \mid \Theta \vdash \Phi \\
\textbf{Rule 1.15.} \quad & \frac{\Xi \mid \Gamma \mid \Theta, R: \text{Rel}(\tau, \tau') \mid \Phi \vdash \psi}{\Xi \mid \Gamma \mid \Theta \mid \Phi \vdash \forall R: \text{Rel}(\tau, \tau'). \psi} \Xi \mid \Gamma \mid \Theta \vdash \Phi \\
\textbf{Rule 1.16.} \quad & \frac{\Xi \mid \Gamma \mid \Theta, S: \text{AdmRel}(\tau, \tau') \mid \Phi \vdash \psi}{\Xi \mid \Gamma \mid \Theta \mid \Phi \vdash \forall S: \text{AdmRel}(\tau, \tau'). \psi} \Xi \mid \Gamma \mid \Theta \vdash \Phi
\end{aligned}$$

Rules for \exists -quantification:

$$\begin{aligned}
\textbf{Rule 1.17.} \quad & \frac{\Xi, \alpha \mid \Gamma \mid \Theta \mid \phi \vdash \psi}{\Xi \mid \Gamma \mid \Theta \mid \exists \alpha: \text{Type}. \phi \vdash \psi} \Xi \mid \Gamma \mid \Theta \vdash \psi \\
\textbf{Rule 1.18.} \quad & \frac{\Xi \mid \Gamma, x: \sigma \mid \Theta \mid \phi \vdash \psi}{\Xi \mid \Gamma \mid \Theta \mid \exists x: \sigma. \phi \vdash \psi} \Xi \mid \Gamma \mid \Theta \vdash \psi \\
\textbf{Rule 1.19.} \quad & \frac{\Xi \mid \Gamma \mid \Theta, R: \text{Rel}(\tau, \tau') \mid \phi \vdash \psi}{\Xi \mid \Gamma \mid \Theta \mid \exists R: \text{Rel}(\tau, \tau'). \phi \vdash \psi} \Xi \mid \Gamma \mid \Theta \vdash \psi \\
\textbf{Rule 1.20.} \quad & \frac{\Xi \mid \Gamma \mid \Theta, S: \text{AdmRel}(\tau, \tau') \mid \phi \vdash \psi}{\Xi \mid \Gamma \mid \Theta \mid \exists S: \text{AdmRel}(\tau, \tau'). \phi \vdash \psi} \Xi \mid \Gamma \mid \Theta \vdash \psi
\end{aligned}$$

External equality implies internal equality:

$$\textbf{Rule 1.21.} \frac{\Xi \mid \Gamma; - \vdash t, u : \sigma \quad \Xi \mid \Gamma; - \vdash t = u}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash t =_{\sigma} u}$$

There are also obvious rules expressing that internal equality is an equivalence relation. We have rules concerning the interpretation of types as relations:

$$\textbf{Rule 1.22.} \frac{\vec{\alpha} \vdash \alpha_i : \text{Type} \quad \Xi \mid \Gamma \mid \Theta \vdash \vec{\rho} : \text{AdmRel}(\vec{\tau}, \vec{\tau}')}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash \alpha_i[\vec{\rho}] \equiv \rho_i}$$

$$\textbf{Rule 1.23.} \frac{\vec{\alpha} \vdash \sigma \multimap \sigma' : \text{Type} \quad \Xi \mid \Gamma \mid \Theta \vdash \vec{\rho} : \text{AdmRel}(\vec{\tau}, \vec{\tau}')}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash (\sigma \multimap \sigma')[\vec{\rho}] \equiv (\sigma[\vec{\rho}] \multimap \sigma'[\vec{\rho}])}$$

$$\textbf{Rule 1.24.} \frac{\vec{\alpha} \vdash \sigma \otimes \sigma' : \text{Type} \quad \Xi \mid \Gamma \mid \Theta \vdash \vec{\rho} : \text{AdmRel}(\vec{\tau}, \vec{\tau}')}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash (\sigma \otimes \sigma')[\vec{\rho}] \equiv (\sigma[\vec{\rho}] \otimes \sigma'[\vec{\rho}])}$$

$$\textbf{Rule 1.25.} \frac{\Xi \mid \Gamma \mid \Theta \vdash \vec{\rho} : \text{AdmRel}(\vec{\tau}, \vec{\tau}')}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash I[\vec{\rho}] \equiv I_{\text{Rel}}}$$

$$\textbf{Rule 1.26.} \frac{\vec{\alpha} \vdash \prod \beta. \sigma(\vec{\alpha}, \beta) : \text{Type} \quad \Xi \mid \Gamma \mid \Theta \vdash \vec{\rho} : \text{AdmRel}(\vec{\tau}, \vec{\tau}')}{\Xi \mid \Gamma \mid \top \vdash (\prod \beta. \sigma(\vec{\alpha}, \beta))[\vec{\rho}] \equiv \forall(\beta, \beta', R : \text{AdmRel}(\beta, \beta')). \sigma[\vec{\rho}, R]}$$

$$\textbf{Rule 1.27.} \frac{\vec{\alpha} \vdash !\sigma : \text{Type} \quad \Xi \mid \Gamma \mid \Theta \vdash \vec{\rho} : \text{AdmRel}(\vec{\tau}, \vec{\tau}')}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash (!\sigma)[\vec{\rho}] \equiv !(\sigma[\vec{\rho}])}$$

If the definable relation ρ is of the form $(x : \sigma, y : \tau). \phi(x, y)$, then $\rho(t, u)$ is equivalent to ϕ with x, y substituted by t, u :

$$\textbf{Rule 1.28.} \frac{\Xi \mid \Gamma, x : \sigma, y : \tau \mid \Theta \vdash \phi : \text{Prop} \quad \Xi \mid \Gamma; - \vdash t : \sigma, u : \tau}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash ((x : \sigma, y : \tau). \phi)(t, u) \sqsubset \phi[t, u/x, y]}$$

$$\textbf{Axiom 1.29.} \frac{}{\Xi \mid \Gamma \mid \Theta \mid \top \vdash (\prod \alpha. (\alpha \rightarrow \alpha) \rightarrow \alpha)(Y, Y)}$$

Given a definable relation ρ we may construct a proposition $\rho(x, y)$. On the other hand, if ϕ is a proposition containing two free variables x and y , then we may construct the definable relation $(x, y). \phi$. The next lemma tells us that these constructions give a correspondence between definable relations and propositions, which is bijective up to provable equivalence in the logic.

Lemma 1.30. *Suppose $\Xi \mid \Gamma, x : \sigma, y : \tau \mid \Theta \vdash \phi$ is a proposition. Then*

$$\Xi \mid \Gamma \mid \Theta \mid \top \vdash ((x : \sigma, y : \tau). \phi)(x, y) \sqsubset \phi$$

Suppose $\Xi \mid \Gamma \mid \Theta \vdash \rho : \text{Rel}(\sigma, \tau)$ is a definable relation, then

$$\Xi \mid \Gamma \mid \Theta \mid \top \vdash \rho \equiv (x : \sigma, y : \tau). \rho(x, y).$$

The substitution axiom above implies the *replacement* rule:

Lemma 1.31.

$$\frac{\Xi \mid \Gamma \mid - \mid \top \vdash t =_{\sigma} t' \quad \Xi \mid \Gamma, x : \sigma; - \vdash u : \tau}{\Xi \mid \Gamma \mid - \mid \top \vdash u[t/x] =_{\tau} u[t'/x]}$$

Proof. Consider the definable relation

$$\rho = (y : \sigma, z : \sigma). u[y/x] =_{\tau} u[z/x].$$

Clearly $\rho(t, t)$ holds, so by substitution $\rho(t, t')$ holds. \square

Lemma 1.32. *Suppose $\Xi \mid \Gamma \mid \Theta \vdash \rho : \text{Rel}(\sigma, \tau), \rho' : \text{Rel}(\sigma', \tau')$ and x, x', y, y' are fresh variables. Then*

$$\Xi \mid \Gamma \mid \Theta \vdash \forall x : \sigma, x' : \sigma', y : \tau, y' : \tau'. \rho(x, y) \wedge \rho'(x', y') \supset \rho \otimes \rho'(x \otimes x', y \otimes y')$$

Proof. Suppose $\rho(x, y) \wedge \rho'(x', y')$ and that $(\rho \multimap \rho' \multimap R)(t, t')$. Then clearly $R(t x x', t' y y')$ and thus, since

$$\text{let } x \otimes x' \text{ be } x \otimes x' \text{ in } t x x' = t x x',$$

we conclude $\rho \otimes \rho'(x \otimes x', y \otimes y')$. \square

Lemma 1.33. *Suppose $\Xi \mid \Gamma \mid \Theta \vdash \rho : \text{Rel}(\sigma, \tau)$ then*

$$\Xi \mid \Gamma \mid \Theta \mid \top \vdash \forall x : \sigma, y : \tau. \rho(x, y) \supset (!\rho)(!x, !y)$$

If $\Xi \mid \Gamma \mid \Theta \vdash \rho : \text{AdmRel}(\sigma, \tau)$ then

$$\Xi \mid \Gamma \mid \Theta \mid \top \vdash \forall x : \sigma, y : \tau. \rho(x, y) \multimap (!\rho)(!x, !y)$$

Proof. The first statement is clear from the definition of $!\rho$. For the right to left implication in the case of ρ being admissible, observe that $(\lambda x : \sigma. x, \lambda x : \tau. x) : \rho \rightarrow \rho$. Since $!x(!\rho)!y$ this implies

$$\rho((\lambda x : \sigma. x)(!x), (\lambda x : \tau. x)(!y)),$$

i.e., $\rho(x, y)$. \square

Recall that in Section 1.2.2 the oconstruction \rightarrow was defined directly on relations, whereas in `PILLY` the type constructor \rightarrow is shorthand for $!(-) \multimap (=)$. The next lemma shows that the relations $\rho \rightarrow \rho'$ and $!\rho \multimap \rho'$ coincide in the case of ρ' being admissible.

Lemma 1.34. *Suppose $\Xi \mid \Gamma \mid \Theta \vdash \rho : \text{Rel}(\sigma, \tau), \rho' : \text{Rel}(\sigma', \tau')$. Then*

$$\Xi \mid \Gamma \mid \Theta \mid \top \vdash \forall f : \sigma \rightarrow \sigma', g : \tau \rightarrow \tau'. (\rho \rightarrow \rho')(f, g) \supset (!\rho \multimap \rho')(f, g).$$

If $\Xi \mid \Gamma \mid \Theta \vdash \rho : \text{Rel}(\sigma, \tau), \rho' : \text{AdmRel}(\sigma', \tau')$ then

$$\Xi \mid \Gamma \mid \Theta \mid \top \vdash (\rho \rightarrow \rho') \equiv (!\rho \multimap \rho').$$

Proof. For the first implication, suppose $(f, g) : !\rho \multimap \rho'$. We must show that if $\rho(x, y)$ then $\rho'(f(!x), g(!y))$, which follows from the assumptions since $!\rho(!x, !y)$.

For the second half, we must show that if ρ' is admissible and $\rho \rightarrow \rho'(f, g)$ and $!\rho(x, y)$ then $\rho'(f(x), g(y))$. But this follows from the definition of $!\rho(x, y)$ when using that ρ' is admissible. \square

1.2.5. *A closure operator for admissible relations.* In this section we present a closure operator on relations giving the least admissible relation containing a given relation. This closure operator will be particularly useful for proving coinduction principles later. Recall from Proposition 1.7 that for ρ any relation and ρ' admissible $\rho \multimap \rho'$ is admissible. This means that for any relation $\rho: \text{Rel}(\sigma, \tau)$,

$$(\forall \alpha, \beta, S: \text{AdmRel}(\alpha, \beta)). (\rho \multimap S) \multimap S$$

is an admissible relation from $\prod \alpha. (\sigma \multimap \alpha) \multimap \alpha$ to $\prod \alpha. (\tau \multimap \alpha) \multimap \alpha$. We define $\Phi(\rho)$ to be the admissible relation obtained by pulling back this relation along the canonical maps $\sigma \multimap \prod \alpha. (\sigma \multimap \alpha) \multimap \alpha$ and $\tau \multimap \prod \alpha. (\tau \multimap \alpha) \multimap \alpha$, i.e. $\Phi(\rho)$ is

$$(x: \sigma, y: \tau). (\forall \alpha, \beta, S: \text{AdmRel}(\alpha, \beta)). \forall f, g. (\rho \multimap S)(f, g) \supset S(f(x), g(y)).$$

Lemma 1.35. *The operator Φ preserves implication of relations and for any relation ρ , $\Phi(\rho)$ is the smallest admissible relation containing ρ , i.e.,*

$$\begin{aligned} \text{if} \quad & \Xi \mid \Gamma \mid \Theta \vdash \rho: \text{Rel}(\sigma, \tau), \rho': \text{AdmRel}(\sigma, \tau) \\ \text{then} \quad & \Xi \mid \Gamma \mid \Theta \mid \top \vdash \forall x: \sigma, y: \tau. \rho(x, y) \supset \rho'(x, y) \\ \text{iff} \quad & \Xi \mid \Gamma \mid \Theta \mid \top \vdash \forall x: \sigma, y: \tau. \Phi(\rho)(x, y) \supset \rho'(x, y) \end{aligned}$$

In later a paper we will show how the programming language Lily gives rise to a model of LAPL. In this concrete model the notion of admissibility is modeled by the $\top\top$ -closed relations, and so the admissible closure operation presented here coincides with $\top\top$ -closure as defined in [BPR00].

Remark 1.36. Lemma 1.35 provides an alternative way of viewing the constructions on relations presented in Section 1.2.2. In fact $!\rho$ is the smallest admissible relation containing all pairs of the form $(!x, !y)$ for $\rho(x, y)$. Likewise $\rho \otimes \rho'$ is the smallest admissible relation containing all pairs $(x \otimes x', y \otimes y')$ with $\rho(x, y) \wedge \rho'(x', y')$, and I_{Rel} is the smallest admissible relation containing (\star, \star) .

1.2.6. *Extensionality and Identity Extension Schemes.* Consider the two **extensionality schemes**:

$$\begin{aligned} & (\forall x: \sigma. t x =_{\tau} u x) \supset t =_{\sigma \multimap \tau} u \\ & (\forall \alpha: \text{Type}. t \alpha =_{\tau} u \alpha) \supset t =_{\prod \alpha: \text{Type}. \tau} u. \end{aligned}$$

These are taken as axioms in [PA93], but we shall not take these as axioms as we would like to be able to talk about models that are not necessarily extensional.

Lemma 1.37. *It is provable in the logic that*

$$\forall f, g: \sigma \rightarrow \tau. (\forall x: \sigma. f(!x) =_{\tau} g(!x)) \supset \forall x: !\sigma. f(x) =_{\tau} g(x).$$

In particular, extensionality implies

$$\forall f, g: \sigma \rightarrow \tau. (\forall x: \sigma. f(!x) =_{\tau} g(!x)) \supset f =_{\sigma \rightarrow \tau} g$$

Proof. The first formula of the theorem is just the statement that $(f, g): eq_{\sigma} \rightarrow eq_{\tau}$ implies $(f, g): !eq_{\sigma} \multimap eq_{\tau}$. The second formula follows from the first. \square

The schema

$$- \mid - \mid - \vdash \forall \vec{\alpha}: \text{Type}. \sigma[eq_{\vec{\alpha}}] \equiv eq_{\sigma(\vec{\alpha})}$$

is called the **identity extension schema**. Here σ ranges over all types, and $eq_{\vec{\alpha}}$ is short notation for $eq_{\alpha_1}, \dots, eq_{\alpha_n}$.

For any type $\beta, \alpha_1, \dots, \alpha_n \vdash \sigma(\beta, \vec{\alpha})$ we can form the **parametricity schema**:

$$- \mid - \mid - \vdash \forall \vec{\alpha} \forall u: (\prod \beta. \sigma). \forall \beta, \beta'. \forall R: \text{AdmRel}(\beta, \beta'). (u \beta) \sigma[R, eq_{\vec{\alpha}}](u \beta'),$$

where, for readability, we have omitted $: \text{Type}$ after β, β' .

We remark that the reason that parametricity and identity extension are formulated as schemas is that we cannot quantify over type constructors.

Proposition 1.38. *The identity extension schema implies the parametricity schema.*

Proof. The identity extension schema tells us that

$$\forall \vec{\alpha} \forall u: (\prod \beta. \sigma). u(\prod \beta. \sigma)[eq_{\vec{\alpha}}]u.$$

Writing out this expression using Rule 1.26 for the relational interpretation of polymorphic types, one obtains the parametricity schema. \square

In the case of second-order lambda-calculus, the parametricity schema implied identity extension for the pure calculus, since it provided the case of polymorphic types in a proof by induction. It is interesting to notice that this does not seem to be the case for PILL_Y , since it seems that we need identity extension to prove for example $eq_{\sigma} \otimes eq_{\tau} \equiv eq_{\sigma \otimes \tau}$.

1.3. Logical Relations Lemma. We end our presentation of Linear Abadi & Plotkin Logic with the logical relations lemma.

Lemma 1.39 (Logical Relations Lemma). *In pure LAPL, for any closed term $- \mid -; - \vdash t: \tau$,*

$$t \tau t.$$

In words, any closed term of closed type, is related to itself in the relational interpretation of the type. More generally, for any open term

$$\vec{\alpha} \mid \vec{x}': \vec{\sigma}'(\alpha); \vec{x}: \vec{\sigma}(\alpha) \vdash t(\vec{\alpha}, \vec{x}', \vec{x}): \tau$$

in the pure calculus; the proposition

$$\begin{aligned} \forall \vec{\alpha}, \vec{\beta}. \forall \vec{R}: \text{AdmRel}(\vec{\alpha}, \vec{\beta}). \forall \vec{x}: \vec{\sigma}(\vec{\alpha}), \vec{y}: \vec{\sigma}(\vec{\beta}). \forall \vec{x}': \vec{\sigma}'(\vec{\alpha}), \vec{y}': \vec{\sigma}'(\vec{\beta}). \\ \vec{x} \vec{\sigma}[\vec{R}] \vec{y} \wedge \vec{x}' \vec{\sigma}'[\vec{R}] \vec{y}' \supset t(\vec{\alpha}, \vec{x}', \vec{x}) \tau[\vec{R}] t(\vec{\beta}, \vec{y}', \vec{y}) \end{aligned}$$

holds in the logic.

A detailed proof of the Logical Relations Lemma can be found in [Møg05a].

$$\begin{aligned}
 \sigma &\cong \prod \alpha. (\sigma \multimap \alpha) \multimap \alpha \\
 \sigma \otimes \tau &\cong \prod \alpha. (\sigma \multimap \tau \multimap \alpha) \multimap \alpha \\
 !\sigma &\cong \prod \alpha. (\sigma \rightarrow \alpha) \multimap \alpha \\
 I &\cong \prod \alpha. \alpha \multimap \alpha \\
 0 &= \prod \alpha. \alpha \\
 1 &= \prod \alpha. \alpha \\
 \sigma + \tau &= \prod \alpha. (\sigma \multimap \alpha) \rightarrow (\tau \multimap \alpha) \rightarrow \alpha \\
 \sigma \times \tau &= \prod \alpha. (\sigma \multimap \alpha) + (\tau \multimap \alpha) \multimap \alpha \\
 \mathbb{N} &= \prod \alpha. (\alpha \multimap \alpha) \rightarrow \alpha \multimap \alpha \\
 \coprod \alpha. \sigma &= \prod \beta. (\prod \alpha. \sigma \multimap \beta) \multimap \beta \\
 \mu \alpha. \sigma &= \prod \alpha. (\sigma \multimap \alpha) \rightarrow \alpha \\
 \nu \alpha. \sigma &= \prod \alpha. !(\alpha \multimap \sigma) \otimes \alpha
 \end{aligned}$$

Figure 5: Types definable using parametricity

2. ENCODING DATATYPES USING PARAMETRICITY

In this section we show how to use the logic to prove correctness of encodings of a large class of data types in PILL_Y using parametricity. These encoding are due to Plotkin, and many of them are listed Figure 5. In Figure 5 there are two sorts of equations. The first four equations are isomorphisms between types already present in PILL_Y . In these cases we shall show that the isomorphisms hold in a category of *linear* maps, where maps are considered equal up to provability in the logic. We shall give a precise definition of this category shortly.

The other type of equation in Figure 5 defines encodings of types not already present in PILL_Y . We shall show correctness of these encodings, by which we mean that they satisfy the usual universal properties with respect to the above mentioned category of linear maps. In the last two encodings, σ is assumed to be a type expression of PILL_Y in which α occurs only positively (see Section 2.7) in which case $\mu \alpha. \sigma$ defines an initial algebra for the functor induced by σ and $\nu \alpha. \sigma$ defines a final coalgebra. We will also discuss reasoning principles for the encoded types.

We will prove that the fixed point combinator Y causes the initial algebras and final coalgebras to coincide — a phenomenon called algebraic compactness. As a special case we have the coincidence of the initial and final object ($0=1$) as can be seen in Figure 5. Following Freyd [Fre90b, Fre90a, Fre91] we show how algebraic compactness implies the existence of general recursive types in Section 2.11.

In the following we shall write “using extensionality” and “using identity extension” to mean that we assume the extensionality schemes and the identity extension schema, respectively.

2.1. A category of linear functions. The precise formulation of correctness of encodings of the datatypes presented in this section will be that they satisfy the usual universal properties. To state this precisely, we introduce for each kind context Ξ the category $\mathbf{LinType}_\Xi$ as follows:

Objects: are types $\Xi \mid -; - \vdash \sigma : \text{Type}$.

Morphisms: $[\Xi \mid -; - \vdash f : \sigma \multimap \tau]$ are equivalence classes of terms of type $\sigma \multimap \tau$; the equivalence relation on these terms being provable equality in LAPL using extensionality and identity extension.

Composition in this category is given by lambda abstraction, i.e. $f : \sigma \multimap \tau$ composed with $g : \omega \multimap \sigma$ yields $\lambda^\circ x : \omega. f(gx)$.

We start by proving that under the assumption of identity extension and extensionality, for all types $\Xi \vdash \sigma : \mathbf{Type}$ we have an isomorphism of objects of $\mathbf{LinType}_\Xi$:

$$\sigma \cong \prod \alpha. (\sigma \multimap \alpha) \multimap \alpha$$

for α not free in σ . We can define terms

$$f : \sigma \multimap \prod \alpha. ((\sigma \multimap \alpha) \multimap \alpha)$$

and

$$g : \prod \alpha. ((\sigma \multimap \alpha) \multimap \alpha) \multimap \sigma$$

by

$$f = \lambda^\circ x : \sigma. \Lambda \alpha. \lambda^\circ h : \sigma \multimap \alpha. h \ x$$

and

$$g = \lambda^\circ x : \prod \alpha. ((\sigma \multimap \alpha) \multimap \alpha). x \ \sigma \ id_\sigma$$

Clearly

$$g(f \ x) = (f \ x) \ \sigma \ id_\sigma = x$$

so $g \circ f = id_\sigma$. Notice that this only involve external equality and thus we did not need extensionality here.

Proposition 2.1. *Using identity extension and extensionality, one may prove that $f \circ g$ is internally equal to the identity.*

Proof. For a term $a : \prod \alpha. (\sigma \multimap \alpha) \multimap \alpha$ we have

$$f \circ g \ a = \Lambda \alpha. \lambda^\circ h : \sigma \multimap \alpha. h(a \ \sigma \ id_\sigma).$$

Using extensionality, it suffices to prove that

$$\Xi, \alpha \mid h : \sigma \multimap \alpha \mid - \vdash h(a \ \sigma \ id_\sigma) =_\alpha a \ \alpha \ h$$

holds in the internal logic.

By the parametricity schema we know that for any admissible relation $\rho : \mathbf{AdmRel}(\tau, \tau')$

$$(a \ \tau)((eq_\sigma \multimap \rho) \multimap \rho)(a \ \tau')$$

If we instantiate this with the admissible relation $\langle h \rangle$, we get

$$(a \ \sigma)((eq_\sigma \multimap \langle h \rangle) \multimap \langle h \rangle)(a \ \alpha)$$

Since $id_\sigma(eq_\sigma \multimap \langle h \rangle)h$ we know that $(a \ \sigma \ id_\sigma)\langle h \rangle(a \ \alpha \ h)$, i.e.,

$$h(a \ \sigma \ id_\sigma) =_\alpha a \ \alpha \ h,$$

as desired. □

This proof may essentially be found in [Bg05].

Intuitively, what happens here is that σ is a subtype of $\prod \alpha. (\sigma \multimap \alpha) \multimap \alpha$, where the inclusion f maps x to application at x . We use parametricity to show that $\prod \alpha. (\sigma \multimap \alpha) \multimap \alpha$ does not contain anything that is not in σ .

2.2. Tensor types. The goal of this section is to prove

$$\sigma \otimes \tau \cong \prod \alpha. (\sigma \multimap \tau \multimap \alpha) \multimap \alpha$$

using identity extension and extensionality, for $\Xi \vdash \sigma : \mathbf{Type}$ and $\Xi \vdash \tau : \mathbf{Type}$ types in the same context. The isomorphism is in the category $\mathbf{LinType}_\Xi$.

This isomorphism leads to the question of whether tensor types are actually superfluous in the language. The answer is yes in the following sense: Call the language without tensor types (and I) t and the language as is T . Then there are transformations $p : T \rightarrow t$ and $i : t \rightarrow T$, i being the inclusion, such that $p \circ i = id_T$ and $i \circ p \cong id_t$. This is all being stated more precisely, not to mention proved, in [MBR05]. In this paper we settle for the isomorphism above.

We can construct terms

$$f : \sigma \otimes \tau \multimap \prod \alpha. (\sigma \multimap \tau \multimap \alpha) \multimap \alpha$$

and

$$g : (\prod \alpha. (\sigma \multimap \tau \multimap \alpha) \multimap \alpha) \multimap \sigma \otimes \tau$$

by

$$f y = \text{let } x \otimes x' : \sigma \otimes \tau \text{ be } y \text{ in } \Lambda \alpha. \lambda^\circ h : \sigma \multimap \tau \multimap \alpha. h x x'$$

and

$$g y = y (\sigma \otimes \tau) \text{ pairing},$$

where the map $\text{pairing} : \sigma \multimap \tau \multimap \sigma \otimes \tau$ is

$$\text{pairing} = \lambda^\circ x : \sigma. \lambda^\circ x' : \tau. x \otimes x'.$$

Let us show that the composition $g \circ f$ is the identity.

$$\begin{aligned} g \circ f y &= g(\text{let } x \otimes x' : \sigma \otimes \tau \text{ be } y \text{ in } \Lambda \alpha. \lambda^\circ h : \sigma \multimap \tau \multimap \alpha. h x x') \\ &= (\text{let } x \otimes x' : \sigma \otimes \tau \text{ be } y \text{ in } \Lambda \alpha. \lambda^\circ h : \sigma \multimap \tau \multimap \alpha. h x x') (\sigma \otimes \tau) \text{ pairing} \\ &= (\Lambda \alpha. \lambda^\circ h : \sigma \multimap \tau \multimap \alpha. \text{let } x \otimes x' : \sigma \otimes \tau \text{ be } y \text{ in } h x x') (\sigma \otimes \tau) \text{ pairing} \\ &= \text{let } x \otimes x' : \sigma \otimes \tau \text{ be } y \text{ in } x \otimes x' \\ &= y. \end{aligned}$$

Proposition 2.2. *Using extensionality and identity extension one may prove that the composition*

$$f \circ g : (\prod \alpha. (\sigma \multimap \tau \multimap \alpha) \multimap \alpha) \multimap (\prod \alpha. (\sigma \multimap \tau \multimap \alpha) \multimap \alpha)$$

is internally equal to the identity.

Proof. We compute

$$\begin{aligned} f \circ g (y) &= f(y (\sigma \otimes \tau) \text{ pairing}) = \\ &\text{let } x \otimes x' : \sigma \otimes \tau \text{ be } (y (\sigma \otimes \tau) \text{ pairing}) \text{ in } \Lambda \alpha. \lambda^\circ h : \sigma \multimap \tau \multimap \alpha. h x x' \end{aligned}$$

Suppose we are given a type α and a map $h : \sigma \multimap \tau \multimap \alpha$. We can define $\phi_h : \sigma \otimes \tau \multimap \alpha$ as

$$\phi_h = \lambda^\circ y : \sigma \otimes \tau. \text{let } x \otimes x' : \sigma \otimes \tau \text{ be } y \text{ in } h x x'.$$

Then $\phi_h(\text{pairing } x \ x') = h \ x \ x'$, which means that $\text{pairing}(eq_\sigma \multimap eq_\tau \multimap \langle \phi_h \rangle)h$. By the parametricity schema

$$\Xi, \alpha \mid h: \sigma \multimap \tau \multimap \alpha, y: \prod \alpha. (\sigma \multimap \tau \multimap \alpha) \multimap \alpha \mid - \mid \top \vdash \\ (y \ \sigma \otimes \tau)((eq_\sigma \multimap eq_\tau \multimap \langle \phi_h \rangle) \multimap \langle \phi_h \rangle)(y \ \alpha)$$

so

$$(y \ \sigma \otimes \tau \text{ pairing}) \langle \phi_h \rangle (y \ \alpha \ h),$$

i.e.,

$$\phi_h(y \ \sigma \otimes \tau \text{ pairing}) =_\alpha y \ \alpha \ h.$$

Writing this out we get

$$\Xi, \alpha \mid h: \sigma \multimap \tau \multimap \alpha, y: \prod \alpha. (\sigma \multimap \tau \multimap \alpha) \multimap \alpha \mid - \mid \top \vdash \\ \text{let } x \otimes x': \sigma \otimes \tau \text{ be } (y \ \sigma \otimes \tau \text{ pairing}) \text{ in } h \ x \ x' =_\alpha y \ \alpha \ h.$$

Using extensionality we get

$$\Lambda \alpha. \lambda^\circ h: \sigma \multimap \tau \multimap \alpha. \text{let } x \otimes x': \sigma \otimes \tau \text{ be } (y \ \sigma \otimes \tau \text{ pairing}) \text{ in } (h \ x \ x') =_\alpha y.$$

This is enough, since by the rules for external equality the left hand side is

$$\text{let } x \otimes x': \sigma \otimes \tau \text{ be } (y \ \sigma \otimes \tau \text{ pairing}) \text{ in } (\Lambda \alpha. \lambda^\circ h: \sigma \multimap \tau \multimap \alpha. h \ x \ x').$$

□

2.3. Unit object. The goal of this section is to prove that identity extension together with extensionality implies

$$I \cong \prod \alpha. \alpha \multimap \alpha.$$

The isomorphism holds in **LinType**_Ξ for all Ξ.

We first define maps $f: I \multimap \prod \alpha. \alpha \multimap \alpha$ and $g: (\prod \alpha. \alpha \multimap \alpha) \multimap I$ as

$$f = \lambda^\circ x: I. \text{let } \star \text{ be } x \text{ in } id, \\ g = \lambda^\circ t: \prod \alpha. \alpha \multimap \alpha. t \ I \ \star,$$

where

$$id = \Lambda \alpha. \lambda^\circ y: \alpha. y.$$

We first notice that

$$\begin{aligned} g(f(x)) &= (\text{let } \star \text{ be } x \text{ in } id) \ I \ \star \\ &= \text{let } \star \text{ be } x \text{ in } (id \ I \ \star) \\ &= \text{let } \star \text{ be } x \text{ in } \star \\ &= x. \end{aligned}$$

Proposition 2.3. *Using identity extension and extensionality, we have that $f \circ g$ is internally equal to the identity on $\prod \alpha. \alpha \multimap \alpha$.*

Proof. First we write out the definition

$$f \circ g = \lambda^\circ t: (\prod \alpha. \alpha \multimap \alpha). \text{let } \star \text{ be } (t \ I \ \star) \text{ in } id.$$

We show that for any $t: \prod \alpha. \alpha \multimap \alpha$, for any type σ , and any $x: \sigma$ we have $f \circ g(t) \sigma x =_\sigma t \sigma x$.

Given σ, x as above, we can define $h: I \multimap \sigma$ as $h = \lambda^\circ z: I. \text{let } \star \text{ be } z \text{ in } x$. Then $\langle h \rangle$ is admissible, so by identity extension

$$(t \ I) (\langle h \rangle \multimap \langle h \rangle) (t \ \sigma).$$

Since $h(\star) = x$ we have $h(t I \star) =_{\sigma} t \sigma x$, and by definition

$$\begin{aligned} h(t I \star) &= \text{let } \star \text{ be } (t I \star) \text{ in } x \\ &= \text{let } \star \text{ be } (t I \star) \text{ in } (id \sigma x) \\ &= (\text{let } \star \text{ be } (t I \star) \text{ in } id) \sigma x \\ &= f \circ g(t) \sigma x. \end{aligned}$$

□

2.4. Initial objects and coproducts. We define

$$0 = \prod \alpha. \alpha$$

For each Ξ this defines a weak initial object in **LinType** $_{\Xi}$, since for any type $\Xi \vdash \sigma$, there exists a term $0_{\sigma}: 0 \multimap \sigma$, defined as

$$\lambda^{\circ} x: 0. x \sigma$$

Proposition 2.4. *Suppose $f: 0 \multimap \sigma$ for some type $\Xi \vdash \sigma$. Using identity extension and extensionality it is provable that $f =_{0 \multimap \sigma} 0_{\sigma}$. Thus, 0 is an initial object in **LinType** $_{\Xi}$ for each Ξ .*

Proof. First notice that for any map $h: \sigma \multimap \tau$, by identity extension $(x \sigma) \langle h \rangle (x \tau)$ for any $x: 0$. Thus, by extensionality, $h \circ 0_{\sigma} =_{0 \multimap \tau} 0_{\tau}$ for any $h: \sigma \multimap \tau$. In particular, for any type σ , the case $h = 0_{\sigma}$ gives us $x 0 \sigma =_{\sigma} x \sigma$, i.e., $0_0 =_{0 \multimap \sigma} id_0$. If $f: 0 \multimap \sigma$, by the above we have $0_{\sigma} =_{0 \multimap \sigma} f \circ 0_0 =_{0 \multimap \sigma} f$ □

Next, suppose $\Xi \vdash \sigma, \tau$ are types in the same context. We define

$$\sigma + \tau = \prod \alpha. (\sigma \multimap \alpha) \rightarrow (\tau \multimap \alpha) \rightarrow \alpha$$

and show under the assumption of identity extension and extensionality that this defines a coproduct of σ and τ in **LinType** $_{\Xi}$.

First define terms $in_{\sigma}: \sigma \multimap \sigma + \tau$, $in_{\tau}: \tau \multimap \sigma + \tau$ as

$$\begin{aligned} in_{\sigma} &= \lambda^{\circ} x: \sigma. \Lambda \alpha. \lambda f: \sigma \multimap \alpha. \lambda g: \tau \multimap \alpha. f(x) \\ in_{\tau} &= \lambda^{\circ} y: \tau. \Lambda \alpha. \lambda f: \sigma \multimap \alpha. \lambda g: \tau \multimap \alpha. g(y) \end{aligned}$$

For any pair of maps $f: \sigma \multimap \omega$, $g: \tau \multimap \omega$ define the copairing $[f, g]: \sigma + \tau \multimap \omega$ as

$$[f, g] = \lambda^{\circ} x: \sigma + \tau. x \omega ! f ! g,$$

then clearly $[f, g](in_{\sigma}(x)) = f(x)$ and $[f, g](in_{\tau}(y)) = g(y)$, and so $\sigma + \tau$ is a weak coproduct of σ and τ in **LinType** $_{\Xi}$. We remark that the copairing constructor can also be defined as a polymorphic term

$$[-, -]: \Lambda \alpha. (\sigma \multimap \alpha) \rightarrow (\tau \multimap \alpha) \rightarrow \sigma + \tau \multimap \alpha$$

of *intuitionistic* function type. Of course we can define an even more general copairing by abstracting σ, τ as well.

Lemma 2.5. *If $h: \omega \multimap \omega'$, $f: \sigma \multimap \omega$ and $g: \tau \multimap \omega$, then using extensionality and identity extension, it is provable that $[h \circ f, h \circ g] =_{\sigma + \tau \multimap \omega'} h \circ [f, g]$.*

Proof. Since

$$\begin{aligned} f(eq_\sigma \multimap \langle h \rangle) h \circ f \\ g(eq_\tau \multimap \langle h \rangle) h \circ g \end{aligned}$$

for any $x: \sigma + \tau$,

$$(x \omega !f !g) \langle h \rangle (x \omega' !(h \circ f) !(h \circ g))$$

by identity extension, i.e., $h([f, g](x)) = [h \circ f, h \circ g](x)$. \square

Lemma 2.6. *Using extensionality and identity extension, $[in_\sigma, in_\tau] =_{\sigma+\tau \multimap \sigma+\tau} id_{\sigma+\tau}$ is provable.*

Proof. Given any $\omega, a: \sigma \multimap \omega, b: \tau \multimap \omega$, we have

$$[a, b]([in_\sigma, in_\tau](x)) =_\omega [[a, b] \circ in_\sigma, [a, b] \circ in_\tau](x) =_\omega [a, b](x)$$

for any $x: \sigma + \tau$. By unfolding the definition of $[a, b]$ in the above equality we get

$$[in_\sigma, in_\tau](x) \omega !a !b =_\omega x \omega !a !b.$$

Since ω, a, b were arbitrary, extensionality (and Lemma 1.37) implies $[in_\sigma, in_\tau](x) =_{\sigma+\tau} x$ for all x . \square

Proposition 2.7. *For any $f: \sigma \multimap \omega, g: \tau \multimap \omega$ and $h: \sigma + \tau \multimap \omega$, if $h \circ in_\sigma =_{\sigma \multimap \omega} f$ and $h \circ in_\tau =_{\tau \multimap \omega} g$, then it is provable using identity extension and extensionality that $h =_{\sigma+\tau \multimap \omega} [f, g]$. Thus $\sigma + \tau$ is a coproduct of σ and τ in $\mathbf{LinType}_\Xi$.*

Proof.

$$[f, g] =_{\sigma+\tau \multimap \omega} [h \circ in_\sigma, h \circ in_\tau] =_{\sigma+\tau \multimap \omega} h \circ [in_\sigma, in_\tau] =_{\sigma+\tau \multimap \omega} h$$

\square

2.5. Terminal objects and products. The initial object 0 is also weakly terminal, since for any type σ ,

$$\Omega_{\sigma \multimap 0} = Y \sigma !id_{\sigma \multimap 0}$$

is a term of type $\sigma \multimap 0$. In fact, using parametricity, 0 can be proved to be terminal.

Proposition 2.8. *Suppose $f, g: \sigma \multimap 0$. Using identity extension and extensionality it is provable that $f =_{\sigma \multimap 0} g$. Thus 0 is a terminal object in $\mathbf{LinType}_\Xi$ for any Ξ .*

Proof. We will prove

$$\forall x, y: 0. x =_0 y$$

which, by extensionality, implies the proposition. Suppose we are given $x, y: 0$. The term

$$\lambda^\circ z: 0. z \multimap \sigma y$$

has type $0 \multimap \sigma$, and thus is equal to 0_σ . This means that $x \sigma =_\sigma x \multimap \sigma y$. Likewise $x \multimap \sigma y =_\sigma y \sigma$, so $x \sigma =_\sigma y \sigma$. Since this holds for all σ , by extensionality $x =_0 y$. \square

Suppose σ, τ are types in the same context Ξ . Define

$$\sigma \times \tau = \prod \alpha. (\sigma \multimap \alpha) + (\tau \multimap \alpha) \multimap \alpha.$$

This defines a weak product in **LinType** $_{\Xi}$ with projections $\pi_{\sigma}: \sigma \times \tau \multimap \sigma$ and $\pi_{\tau}: \sigma \times \tau \multimap \tau$ defined as

$$\begin{aligned} \pi_{\sigma} &= \lambda^{\circ} x: \sigma \times \tau. x \sigma (in_{\sigma \multimap \sigma} id_{\sigma}) \\ \pi_{\tau} &= \lambda^{\circ} x: \sigma \times \tau. x \tau (in_{\tau \multimap \tau} id_{\tau}) \end{aligned}$$

The pairing of terms $f: \omega \multimap \sigma$ and $g: \omega \multimap \tau$ is $\langle f, g \rangle: \omega \multimap \sigma \times \tau$ defined as

$$\langle f, g \rangle = \lambda^{\circ} x: \omega. \Lambda \alpha. \lambda^{\circ} h: (\sigma \multimap \alpha) + (\tau \multimap \alpha). [\lambda^{\circ} z: \sigma \multimap \alpha. z \circ f, \lambda^{\circ} z: \tau \multimap \alpha. z \circ g] h x$$

Then

$$\pi_{\sigma}(\langle f, g \rangle(x)) = \langle f, g \rangle(x) \sigma (in_{\sigma \multimap \sigma} id_{\sigma}) = (\lambda^{\circ} z: \sigma \multimap \sigma. z \circ f) id_{\sigma} x = f(x)$$

and so $\pi_{\sigma} \circ \langle f, g \rangle = f$ and likewise $\pi_{\tau} \circ \langle f, g \rangle = g$ proving that $\sigma \times \tau$ defines a weak product.

Lemma 2.9. *Using identity extension and extensionality it is provable that for any $f: \omega \multimap \sigma$, $g: \omega \multimap \tau$, $k: \omega' \multimap \omega$,*

$$\langle f, g \rangle \circ k =_{\omega' \multimap \sigma \times \tau} \langle f \circ k, g \circ k \rangle$$

Proof. The lemma is easily proved by the following direct computation using properties of coproducts established above. The notation $(- \circ k)$ below denotes the term $\lambda^{\circ} y: \omega \multimap \alpha. y \circ k$ of type $(\omega \multimap \alpha) \multimap \omega' \multimap \alpha$.

$$\begin{aligned} & \langle f \circ k, g \circ k \rangle(x) \\ &=_{\sigma \times \tau} \Lambda \alpha. \lambda^{\circ} h: (\sigma \multimap \alpha) + (\tau \multimap \alpha). [\lambda^{\circ} z: \sigma \multimap \alpha. z \circ f \circ k, \lambda^{\circ} z: \tau \multimap \alpha. z \circ g \circ k] h x \\ &=_{\sigma \times \tau} \Lambda \alpha. \lambda^{\circ} h. [(- \circ k) \circ (\lambda^{\circ} z: \sigma \multimap \alpha. z \circ f), (- \circ k) \circ (\lambda^{\circ} z: \tau \multimap \alpha. z \circ g)] h x \\ &=_{\sigma \times \tau} \Lambda \alpha. \lambda^{\circ} h. (- \circ k) \circ [(\lambda^{\circ} z: \sigma \multimap \alpha. z \circ f), (\lambda^{\circ} z: \tau \multimap \alpha. z \circ g)] h x \\ &=_{\sigma \times \tau} \Lambda \alpha. \lambda^{\circ} h. [(\lambda^{\circ} z: \sigma \multimap \alpha. z \circ f), (\lambda^{\circ} z: \tau \multimap \alpha. z \circ g)] h (k(x)) \\ &=_{\sigma \times \tau} \langle f, g \rangle \circ k(x) \end{aligned}$$

□

Lemma 2.10. *Identity extension and extensionality implies that $\langle \pi_{\sigma}, \pi_{\tau} \rangle =_{\sigma \times \tau \multimap \sigma \times \tau} id_{\sigma \times \tau}$.*

Proof. We must show that for any $x: \sigma \times \tau$, any α and any $h: (\sigma \multimap \alpha) + (\tau \multimap \alpha)$

$$[\lambda^{\circ} z: \sigma \multimap \alpha. z \circ \pi_{\sigma}, \lambda^{\circ} z: \tau \multimap \alpha. z \circ \pi_{\tau}] h x =_{\alpha} x \alpha h$$

In fact, since we are dealing with coproducts, it suffices to show that for any $l: \sigma \multimap \alpha$ and $k: \tau \multimap \alpha$

$$\begin{aligned} l(\pi_{\sigma}(x)) &=_{\alpha} x \alpha (in_{\sigma \multimap \alpha} l) \\ k(\pi_{\tau}(x)) &=_{\alpha} x \alpha (in_{\tau \multimap \alpha} k) \end{aligned}$$

We just prove the first of these equations. Since

$$id_{\sigma}(eq_{\sigma} \multimap \langle l \rangle) l$$

by parametricity of a polymorphic version of in ,

$$in_{\sigma \multimap \sigma}(id_{\sigma})(eq_{\sigma} \multimap \langle l \rangle) + (eq_{\tau} \multimap \langle l \rangle) in_{\sigma \multimap \alpha}(l)$$

and so by parametricity of $x: \sigma \times \tau$

$$x \sigma (in_{\sigma \multimap \sigma} id_{\sigma}) \langle l \rangle x \alpha (in_{\sigma \multimap \alpha} l)$$

i.e.

$$\pi_{\sigma}(x) \langle l \rangle x \alpha (in_{\sigma \multimap \alpha} l)$$

as desired. □

Proposition 2.11. *Suppose $h: \omega \multimap \sigma \times \tau$ is such that $\pi_\sigma \circ h =_{\omega \multimap \sigma} f$ and $\pi_\tau \circ h =_{\omega \multimap \tau} g$ then it is provable using identity extension and extensionality that $h =_{\omega \multimap \sigma \times \tau} \langle f, g \rangle$. Thus $\sigma \times \tau$ is a product of σ and τ in $\mathbf{LinType}_\Xi$.*

Proof.

$$h =_{\omega \multimap \sigma \times \tau} \langle \pi_\sigma, \pi_\tau \rangle \circ h =_{\omega \multimap \sigma \times \tau} \langle \pi_\sigma \circ h, \pi_\tau \circ h \rangle =_{\omega \multimap \sigma \times \tau} \langle f, g \rangle.$$

□

2.6. Natural Numbers. We define the type of natural numbers as

$$\mathbb{N} = \prod \alpha. (\alpha \multimap \alpha) \rightarrow \alpha \multimap \alpha.$$

We further define terms $0: \mathbb{N}$, $s: \mathbb{N} \multimap \mathbb{N}$ as

$$0 = \Lambda \alpha. \lambda f: \alpha \multimap \alpha. \lambda^\circ x: \alpha. x, \quad s = \lambda^\circ y: \mathbb{N}. \Lambda \alpha. \lambda f: \alpha \multimap \alpha. \lambda^\circ x: \alpha. f(y \alpha ! f x)$$

and prove that $(\mathbb{N}, 0, s)$ is a weak natural numbers object in each $\mathbf{LinType}_\Xi$, and, using parametricity and extensionality, an honest natural numbers object.

Suppose we are given a type σ , a term $a: \sigma$ and a morphism $b: \sigma \multimap \sigma$. We can then define $h: \mathbb{N} \multimap \sigma$ as $h(y) = y \sigma ! b a$. Then clearly $h(0) = a$, and $h(sx) = b(x \sigma ! b a) = b(h(x))$, so $(\mathbb{N}, 0, s)$ is a weak natural numbers object.

We can express the weak natural numbers object property as: for all a, b , there exists an h such that

$$\begin{array}{ccccc} I & \xrightarrow{0} & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \\ & \searrow a & \downarrow h & & \downarrow h \\ & & \sigma & \xrightarrow{b} & \sigma \end{array}$$

commutes.

Lemma 2.12. *Identity Extension and extensionality implies*

$$\forall x: \mathbb{N}. x \mathbb{N} ! s 0 =_{\mathbb{N}} x$$

Proof. Suppose we are given σ, a, b and define h as above. Since $b \circ h = h \circ s$ and $h 0 = a$, we have $s(\langle h \rangle \multimap \langle h \rangle) b$ and $0 \langle h \rangle a$, by parametricity of x , $(x \mathbb{N} ! s 0) \langle h \rangle (x \sigma ! b a)$, i.e.,

$$(x \mathbb{N} ! s 0) \sigma ! b a =_{\sigma} x \sigma ! b a.$$

Letting σ range over all types and a, b over all terms, using extensionality and Lemma 1.37, we have

$$x \mathbb{N} ! s 0 =_{\mathbb{N}} x,$$

as desired. □

We can now prove that \mathbb{N} is a natural numbers object in each $\mathbf{LinType}_\Xi$.

Lemma 2.13. *Assuming identity extension and extensionality, given σ, a, b , the map h defined as above is up to internal equality the unique h' such that $h'(0) = a$, $h'(sx) = b(h'x)$.*

Proof. Suppose h' satisfies the requirements of the lemma. Then $s(\langle h' \rangle \multimap \langle h' \rangle) b$ and $0 \langle h' \rangle a$ (this is just a reformulation of the requirements), so for arbitrary $x: \mathbb{N}$, by parametricity of x ,

$$x \sigma ! b a =_{\sigma} h'(x \mathbb{N} ! s 0) =_{\sigma} h'(x).$$

Thus, by extensionality, $h' =_{\mathbb{N} \multimap \sigma} h$. □

2.6.1. *Induction principle.* The parametricity principle for the natural numbers implies, that if $R: \text{AdmRel}(\mathbb{N}, \mathbb{N})$, and $x: \mathbb{N}$, then

$$(x \mathbb{N})((R \multimap R) \rightarrow R \multimap R)(x \mathbb{N}).$$

So if $s(R \multimap R)s$ and $R(0, 0)$, then

$$(x \mathbb{N} !s 0)R(x \mathbb{N} !s 0).$$

By Lemma 2.12, $x \mathbb{N} !s 0 =_{\mathbb{N}} x$, so we can conclude that $R(x, x)$. If ϕ is a proposition on \mathbb{N} such that $(x: \mathbb{N}, y: \mathbb{N}). \phi(x)$ is admissible, then from parametricity we obtain the usual induction principle

$$(\phi(0) \wedge \forall x: \mathbb{N}. \phi(x) \supset \phi(s(x))) \supset \forall x: \mathbb{N}. \phi(x).$$

2.7. Types as functors.

Definition 2.14. We say that $\vec{\alpha} \vdash \sigma: \text{Type}$ is an inductively constructed type, if it can be constructed from free variables $\vec{\alpha}$ and closed types using the type constructors of PILL_Y , i.e., $\multimap, \otimes, I, !$ and $\prod \alpha..$

For example, all types of pure PILL_Y are inductively defined, and if σ is a closed type then $\prod \alpha. \sigma \times \alpha$ is an inductively constructed type. However, some models may contain types that are not inductively constructed! For example, in syntactical models, any basic open type, such as the type $\alpha \vdash \text{lists}(\alpha)$ is not inductively constructed.

We define positive and negative occurrences of free type variables in inductively defined types as usual. The type variable α occurs positive in the type α and the positive occurrences of a type variable α in $\sigma \multimap \tau$ are the positive occurrences of α in τ and the negative in σ . The negative occurrences of α in $\sigma \multimap \tau$ are the positive in σ and the negative in τ . The positive and negative occurrences of α in $\prod \beta. \sigma$ are the positive and negative occurrences in σ for $\alpha \neq \beta$. The rest of the type constructors preserve positive and negative occurrences of type variables.

If $\sigma(\alpha, \beta)$ is an inductively defined type in which the free type variable α appears only negatively and the free type variable β appears only positively, then we can consider σ as a functor $\mathbf{LinType}_{\Xi}^{\text{op}} \times \mathbf{LinType}_{\Xi} \rightarrow \mathbf{LinType}_{\Xi}$ for each Ξ by defining the term

$$M_{\sigma(\alpha, \beta)}: \prod \alpha, \beta, \alpha', \beta'. (\alpha' \multimap \alpha) \rightarrow (\beta \multimap \beta') \rightarrow \sigma(\alpha, \beta) \multimap \sigma(\alpha', \beta'),$$

which behaves as the morphism part of a functor, i.e., it respects composition and preserves identities. We define $M_{\sigma(\alpha, \beta)}$ by structural induction on σ . This construction immediately generalizes to types with fewer or more than two free type variables, each of which appear only positively or only negatively. This idea of the functorial interpretation of types being representable by polymorphic terms has also been used in second order lambda calculus (see e.g. [RP90]).

For the base case of the induction, if $\sigma(\alpha, \beta) = \beta$, define

$$M_{\beta} = \Lambda \alpha, \beta, \alpha', \beta'. \lambda f, g. g.$$

In the case $\sigma(\beta, \alpha) \multimap \tau(\alpha, \beta)$ we define the term

$$\prod \alpha, \beta, \alpha', \beta'. (\alpha' \multimap \alpha) \rightarrow (\beta \multimap \beta') \rightarrow (M_{\sigma(\beta, \alpha) \multimap \tau(\alpha, \beta)}: \sigma(\beta, \alpha) \multimap \tau(\alpha, \beta)) \multimap \sigma(\beta', \alpha') \multimap \tau(\alpha', \beta')$$

by

$$\begin{aligned} M_{\sigma(\beta, \alpha) \multimap \tau(\alpha, \beta)} &= \Lambda \alpha, \beta, \alpha', \beta'. \lambda f, g. \\ \lambda^\circ h: \sigma(\beta, \alpha) \multimap \tau(\alpha, \beta). (M_\tau \alpha \beta \alpha' \beta' f g) \circ h \circ (M_\sigma \beta' \alpha' \beta \alpha g f). \end{aligned}$$

For bang types, we define:

$$\begin{aligned} M_{! \sigma(\alpha, \beta)} &= \Lambda \alpha, \beta, \alpha', \beta'. \lambda f: \alpha' \multimap \alpha. \lambda g: \beta \multimap \beta'. \lambda^\circ x: !\sigma(\alpha, \beta). \\ &\text{let } !y \text{ be } x \text{ in } !(M_{\sigma(\alpha, \beta)} \alpha \beta \alpha' \beta' f g y). \end{aligned}$$

For tensor types, we define:

$$\begin{aligned} M_{\sigma(\alpha, \beta) \otimes \tau(\alpha, \beta)} &= \Lambda \alpha, \beta, \alpha', \beta'. \lambda f, g. \lambda^\circ z: \sigma(\alpha, \beta) \otimes \tau(\alpha, \beta). \\ &\text{let } x \otimes y: \sigma(\alpha, \beta) \otimes \tau(\alpha, \beta) \text{ be } z \text{ in } (M_\sigma \alpha \beta \alpha' \beta' f g x) \otimes (M_\tau \alpha \beta \alpha' \beta' f g y). \end{aligned}$$

The last case is the case of polymorphic types:

$$\begin{aligned} M_{\prod \omega. \sigma(\alpha, \beta)} &= \Lambda \alpha, \beta, \alpha', \beta'. \lambda f, g. \lambda^\circ z: \prod \omega. \sigma(\alpha, \beta). \\ &\Lambda \omega: \text{Type}. M_{\sigma(\alpha, \beta)} \alpha \beta \alpha' \beta' f g (z \omega). \end{aligned}$$

Lemma 2.15. *The term M_σ respects composition and preserves identities, i.e., for $f: \alpha' \multimap \alpha$, $f': \alpha'' \multimap \alpha'$, $g: \beta \multimap \beta'$, and $g': \beta' \multimap \beta''$,*

- $M_{\sigma(\alpha, \beta)} \alpha \beta \alpha'' \beta'' ! (f \circ f') ! (g' \circ g) = (M_{\sigma(\alpha, \beta)} \alpha' \beta' \alpha'' \beta'' ! f' ! g') \circ (M_{\sigma(\alpha, \beta)} \alpha \beta \alpha' \beta' ! f ! g),$
- $M_{\sigma(\alpha, \beta)} \alpha \beta \alpha \beta ! id_\alpha ! id_\beta = id_{\sigma(\alpha, \beta)}.$

Proof. The proof proceeds by induction over the structure of σ , and most of it is the same as in [PA93], except the case of tensor-types and $!$. These cases are essentially proved in [Bar97]. \square

Notice that in the proof of Lemma 2.15 we do not need parametricity. Suppose

$$\Xi \mid -; - \vdash f: \alpha' \multimap \alpha, g: \beta \multimap \beta'.$$

We shall write $\sigma(f, g)$ for

$$M_{\sigma(\alpha, \beta)} \alpha \beta \alpha' \beta' ! f ! g.$$

The type of $\sigma(f, g)$ is $\sigma(\alpha, \beta) \multimap \sigma(\alpha', \beta')$. Notice that we apply M to $!f, !g$, since M is of intuitionistic function type (\rightarrow instead of \multimap). By the previous lemma, σ defines a bifunctor $\mathbf{LinType}_\Xi^{op} \times \mathbf{LinType}_\Xi \rightarrow \mathbf{LinType}_\Xi$ for each Ξ .

First we consider this in the case of only one argument:

Lemma 2.16 (Graph lemma). *Assuming identity extension, for any type $\alpha \vdash \sigma$ with α occurring only positively and any map $f: \tau \multimap \tau'$*

$$\sigma[\langle f \rangle] \equiv \langle \sigma(f) \rangle.$$

Likewise, suppose $\alpha \vdash \sigma'$ is a type with α only occurring negatively. Then identity extension implies

$$\sigma[\langle f \rangle] \equiv \langle \sigma(f) \rangle^{op},$$

where $\langle \sigma(f) \rangle^{op}$ is $(x: \sigma(\tau), y: \sigma(\tau')). \langle \sigma(f) \rangle(y, x).$

Proof. We will only prove the first half of the lemma; the other half is proved the same way. Since α occurs only positively in σ , we will assume for readability that M_σ has type $\prod \alpha, \beta. (\alpha \multimap \beta) \rightarrow \sigma(\alpha) \multimap \sigma(\beta).$

By parametricity of M_σ , we have, for any pair of admissible relations $\rho: \text{AdmRel}(\alpha, \alpha')$ and $\rho': \text{AdmRel}(\beta, \beta'),$

$$(M_\sigma \alpha \beta)((\rho \multimap \rho') \rightarrow (\sigma[\rho] \multimap \sigma[\rho']))(M_\sigma \alpha' \beta'). \quad (2.1)$$

Let $f : \tau \multimap \tau'$ be arbitrary. If we instantiate (2.1) with $\rho = eq_\tau$ and $\rho' = \langle f \rangle$, we get

$$(M_\sigma \tau \tau)((eq_\tau \multimap \langle f \rangle) \rightarrow (eq_{\sigma(\tau)} \multimap \sigma[\langle f \rangle]))(M_\sigma \tau \tau'),$$

using the identity extension schema. Since $id_\tau(eq_\tau \multimap \langle f \rangle)f$,

$$!id_\tau!(eq_\tau \multimap \langle f \rangle)!f,$$

and using $M_\sigma \tau \tau' !f = \sigma(f)$ we get

$$id_{\sigma(\tau)}(eq_{\sigma(\tau)} \multimap \sigma[\langle f \rangle])\sigma(f),$$

i.e.,

$$\forall x : \sigma(\tau). x(\sigma[\langle f \rangle])(\sigma(f)x).$$

We have thus proved $\langle \sigma(f) \rangle$ implies $\sigma[\langle f \rangle]$.

To prove the other direction, instantiate (2.1) with the admissible relations $\rho = \langle f \rangle$, $\rho' = eq_{\tau'}$ for $f : \tau \multimap \tau'$. Since $f(\langle f \rangle \multimap eq_{\tau'})id_{\tau'}$,

$$\sigma(f)(\sigma[\langle f \rangle] \rightarrow eq_{\sigma(\tau')})id_{\sigma(\tau')}.$$

So for any $x : \sigma(\tau)$ and $y : \sigma(\tau')$ we have $x(\sigma[\langle f \rangle])y$ implies $\sigma(f)x =_{\sigma(\tau')} y$. This just means that $\sigma[\langle f \rangle]$ implies $\langle \sigma(f) \rangle$. \square

2.8. Existential types. In this section we consider existential or sum types. If $\Xi, \alpha \vdash \sigma$ is a type, we define the type $\Xi \vdash \coprod \alpha. \sigma$ as

$$\coprod \alpha. \sigma = \prod \beta. (\prod \alpha. \sigma \multimap \beta) \multimap \beta$$

In fact, this defines a functor

$$\mathbf{LinType}_{\Xi, \alpha} \rightarrow \mathbf{LinType}_\Xi$$

with functorial action as defined in Section 2.7. In this section we show that this functor is left adjoint to the weakening functor

$$\mathbf{LinType}_\Xi \rightarrow \mathbf{LinType}_{\Xi, \alpha}$$

mapping a type $\Xi \vdash \sigma$ to $\Xi, \alpha \vdash \sigma$. In other words, we show that for any type $\Xi \vdash \tau$, there is a one-to-one correspondence between terms $\Xi \vdash t : (\coprod \alpha. \sigma) \multimap \tau$ and terms $\Xi, \alpha \vdash \sigma \multimap \tau$ if we consider terms up to internal equality provable using identity extension and extensionality.

First define the term

$$pack : \prod \alpha. (\sigma \multimap \prod \alpha. \sigma)$$

as $\Lambda \alpha. \lambda^\circ x : \sigma. \Lambda \beta. \lambda^\circ f : \prod \alpha. (\sigma \multimap \beta). f \alpha x$. The correspondence is as follows. Suppose first $\Xi, \alpha \vdash t : \sigma \multimap \tau$. Then $\Xi \vdash \hat{t} : (\prod \alpha. \sigma) \multimap \tau$ is $\lambda^\circ x : (\prod \alpha. \sigma). x \tau (\Lambda \alpha. t)$. If $\Xi \vdash s : (\prod \alpha. \sigma) \multimap \tau$ then $\Xi, \alpha \vdash \tilde{s} : \sigma \multimap \tau$ is defined to be $\lambda x : \sigma. s(pack \alpha x)$.

Now, suppose we start with a term $\Xi, \alpha \vdash t : \sigma \multimap \tau$ then

$$\begin{aligned} \tilde{\hat{t}} &= \lambda^\circ x : \sigma. (\lambda^\circ y : \prod \alpha. \sigma. y \tau (\Lambda \alpha. t)) (pack \alpha x) \\ &= \lambda^\circ x : \sigma. pack \alpha x \tau (\Lambda \alpha. t) \\ &= \lambda^\circ x : \sigma. (\Lambda \alpha. t) \alpha x \\ &= t. \end{aligned}$$

It remains to prove that $\hat{\tilde{s}}$ is equal to s for any $\Xi \vdash s : (\prod \alpha. \sigma) \multimap \tau$. For this we need to use identity extension.

Lemma 2.17. *Suppose $x: \prod \alpha. \sigma$, τ, τ' are types and $f: \tau \multimap \tau', g: \prod \alpha. \sigma \multimap \tau$. Then using identity extension and extensionality,*

$$x \tau' (\Lambda \alpha. f \circ (g \alpha)) =_{\tau'} f (x \tau g)$$

Proof. Using identity extension on g it is easy to see that $g(\prod \alpha. \sigma \multimap \langle f \rangle) \Lambda \alpha. f \circ (g \alpha)$. If $x: \prod \alpha. \sigma$ then by identity extension

$$x \tau g \langle f \rangle x \tau' (\Lambda \alpha. f \circ (g \alpha))$$

which is what we needed to prove. \square

Lemma 2.18. *It is provable using identity extension and extensionality that*

$$\forall x: (\prod \alpha. \sigma). x \prod \alpha. \sigma \text{ pack} =_{\prod \alpha. \sigma} x$$

Proof. Suppose we are given β and $f: \prod \alpha. \sigma \multimap \beta$. We show that

$$x \beta f =_{\beta} x (\prod \alpha. \sigma) \text{ pack } \beta f$$

Define $f' = \lambda^\circ x: (\prod \alpha. \sigma) x \beta f$ of type $(\prod \alpha. \sigma) \multimap \beta$. By Lemma 2.17

$$x \beta (\Lambda \alpha. f' \circ (\text{pack } \alpha)) =_{\beta} f' (x \prod \alpha. \sigma \text{ pack}) =_{\beta} x \prod \alpha. \sigma \text{ pack } \beta f$$

so we just need to show that $\Lambda \alpha. f' \circ (\text{pack } \alpha)$ is internally equal to f . But

$$\Lambda \alpha. f' \circ (\text{pack } \alpha) \alpha y =_{\beta} f' (\text{pack } \alpha y) =_{\beta} \text{pack } \alpha y \beta f =_{\beta} f \alpha y.$$

\square

Proposition 2.19. *Suppose $\Xi \vdash s: (\prod \alpha. \sigma) \multimap \tau$. It is provable using identity extension and extensionality that \hat{s} is internally equal to s .*

Proof.

$$\hat{s}(x) =_{\tau} x \tau (\Lambda \alpha. \lambda^\circ x': \sigma. s (\text{pack } \alpha x')) =_{\tau} s (x \prod \alpha. \sigma \text{ pack}) =_{\tau} s x$$

where for the second equality we have used Lemma 2.17. \square

Parametricity induces the following reasoning principle for existential types.

Proposition 2.20. *For $x, y: \prod \alpha. \sigma(\alpha)$ the following is equivalent to internal equality of x and y .*

$$\exists \alpha, \beta, R: \text{AdmRel}(\alpha, \beta), x': \sigma(\alpha), y': \sigma(\beta). x = \text{pack } \alpha x' \wedge y = \text{pack } \beta y' \wedge \sigma[R](x', y').$$

As a special case we get the following principle:

$$\forall x: \prod \alpha. \sigma(\alpha). \exists \alpha, x': \sigma(\alpha). x =_{\prod \alpha. \sigma(\alpha)} \text{pack } \alpha x'$$

Proof. Let us for simplicity write χ for

$$(x, y). \exists \alpha, \beta, R: \text{AdmRel}(\alpha, \beta), x': \sigma(\alpha), y': \sigma(\beta). x = \text{pack } \alpha x' \wedge y = \text{pack } \beta y' \wedge \sigma[R](x', y').$$

We now prove that, for any pair of types τ, τ' , any admissible relation $S: \text{AdmRel}(\tau, \tau')$, and any pair of maps t, t' we have

$$(t, t'): eq_{\prod \alpha. \sigma} \multimap S$$

iff

$$(t, t'): \chi \multimap S$$

and the two implications of the first part of the proposition follows from the case of t, t' both being identity and taking S to be respectively χ and $eq_{\prod \alpha. \sigma}$.

First notice that

$$\frac{\Xi \mid x, y \vdash \chi(x, y) \supset S(t(x), t'(y))}{\frac{\Xi, \alpha, \beta \mid R: \text{AdmRel}(\alpha, \beta) \mid x, y, x', y' \vdash \sigma[R](x', y') \supset S(t(\text{pack } \alpha \ x'), t'(\text{pack } \beta \ y'))}{\Xi, \alpha, \beta \mid R: \text{AdmRel}(\alpha, \beta) \mid x', y' \vdash \sigma[R](x', y') \supset S(\tilde{t}(x'), \tilde{t}'(y'))}}$$

so it suffices to show that

$$\frac{\Xi \mid x, y \vdash x =_{\prod \alpha. \sigma(\alpha)} y \supset S(t(x), t'(y))}{\Xi, \alpha, \beta \mid R: \text{AdmRel}(\alpha, \beta) \mid x', y' \vdash \sigma[R](x', y') \supset S(\tilde{t}(x'), \tilde{t}'(y'))}$$

i.e., that (t, t') preserve relations iff (\tilde{t}, \tilde{t}') do.

First assume (t, t') preserve relations. By parametricity of *pack*,

$$(\text{pack } \alpha, \text{pack } \beta) : \sigma[R] \multimap eq,$$

and so since $\tilde{t} = t \circ (\text{pack } \alpha)$ and $\tilde{t}' = t' \circ (\text{pack } \beta)$ the pair (\tilde{t}, \tilde{t}') preserve relations. On the other hand, if (\tilde{t}, \tilde{t}') preserve relations then

$$(\Lambda \alpha. \tilde{t}, \Lambda \beta. \tilde{t}') : \forall \alpha, \beta, R: \text{AdmRel}(\alpha, \beta). \sigma[R] \multimap S,$$

and so by parametricity, if $eq_{\prod \alpha. \sigma(\alpha)}(x, y)$ then

$$(t(x), t'(y)) = (x \prod \alpha. \sigma(\alpha) (\Lambda \alpha. \tilde{t}), y \prod \alpha. \sigma(\alpha) (\Lambda \beta. \tilde{t}')) \in S$$

□

2.9. Initial algebras. Suppose $\alpha \vdash \sigma : \text{Type}$ is an inductively constructed type in which α occurs only positively. As we have seen earlier, such a type induces a functor

$$\mathbf{LinType}_\Xi \rightarrow \mathbf{LinType}_\Xi$$

for each Ξ . We aim to define an initial algebra for this type.

Define the closed type

$$\mu \alpha. \sigma(\alpha) = \prod \alpha. (\sigma(\alpha) \multimap \alpha) \rightarrow \alpha,$$

and define

$$fold : \prod \alpha. (\sigma(\alpha) \multimap \alpha) \rightarrow (\mu \alpha. \sigma(\alpha) \multimap \alpha)$$

as

$$fold = \Lambda \alpha. \lambda f : \sigma(\alpha) \multimap \alpha. \lambda^\circ u : \mu \alpha. \sigma(\alpha). u \ \alpha \ !f,$$

and

$$in : \sigma(\mu \alpha. \sigma(\alpha)) \multimap \mu \alpha. \sigma(\alpha)$$

as

$$in \ z = \Lambda \alpha. \lambda f : \sigma(\alpha) \multimap \alpha. f(\sigma(fold \ \alpha \ !f) \ z).$$

Lemma 2.21. *For any algebra $f : \sigma(\tau) \multimap \tau$, $fold \tau \ !f$ is a map of algebras from $(\mu \alpha. \sigma(\alpha), in)$ to (τ, f) , i.e., the diagram*

$$\begin{array}{ccc} \sigma(\mu \alpha. \sigma(\alpha)) & \xrightarrow{in} & \mu \alpha. \sigma(\alpha) \\ \sigma(fold \ \tau \ !f) \downarrow & & \downarrow fold \ \tau \ !f \\ \sigma(\tau) & \xrightarrow{f} & \tau \end{array}$$

commutes.

Proof. For $x: \sigma(\mu\alpha.\sigma(\alpha))$

$$(fold\ \tau\ !f) \circ in\ x = in\ x\ \tau\ !f = f(\sigma(fold\ \tau\ !f)\ x),$$

as desired. \square

In words we have shown that *in* defines a weakly initial algebra for the functor defined by σ in **LinType** $_{\Xi}$ for each Ξ . Notice that parametricity was not needed in this proof.

Lemma 2.22. *Suppose $\Xi \mid \Gamma; - \vdash f: \sigma(\tau) \multimap \tau$ and $\Xi \mid \Gamma; - \vdash g: \sigma(\omega) \multimap \omega$ are algebras for σ , and $\Xi \mid \Gamma; - \vdash h: \tau \multimap \omega$ is a map of algebras, i.e., $h\ f = g\ \sigma(h)$. Then, assuming identity extension and extensionality,*

$$h \circ (fold\ \tau\ !f) =_{\mu\alpha.\sigma(\alpha) \multimap \omega} fold\ \omega\ !g.$$

Proof. Since h is a map of algebras

$$f(\langle \sigma(h) \rangle \multimap \langle h \rangle)g,$$

so by the Graph Lemma (2.16)

$$f(\sigma[\langle h \rangle] \multimap \langle h \rangle)g$$

and by Lemma 1.33

$$!f(!(\sigma[\langle h \rangle] \multimap \langle h \rangle))!g.$$

Clearly $(fold, fold) \in eq_{\prod \alpha. (\sigma(\alpha) \multimap \alpha) \rightarrow (\mu\alpha.\sigma(\alpha) \multimap \alpha)}$, and thus, by identity extension,

$$(fold, fold) \in \prod \alpha. (\sigma(\alpha) \multimap \alpha) \rightarrow (\beta \multimap \alpha)[eq_{\mu\alpha.\sigma(\alpha)}/\beta],$$

so for any $x: \mu\alpha.\sigma(\alpha)$,

$$(fold\ \tau\ !f\ x)\langle h \rangle(fold\ \omega\ !g\ x),$$

i.e.,

$$h \circ (fold\ \tau\ !f) =_{\mu\alpha.\sigma(\alpha) \multimap \omega} fold\ \omega\ !g,$$

as desired. \square

Lemma 2.23. *Using identity extension and extensionality,*

$$fold\ \mu\alpha.\sigma(\alpha)\ !in =_{\mu\alpha.\sigma(\alpha) \multimap \mu\alpha.\sigma(\alpha)} id_{\mu\alpha.\sigma(\alpha)}.$$

Proof. By Lemma 2.22 we know that for any type τ , $f: \sigma(\tau) \multimap \tau$ and $u: \mu\alpha.\sigma(\alpha)$

$$(fold\ \tau\ !f) \circ (fold\ \mu\alpha.\sigma(\alpha)\ !in)\ u =_{\tau} fold\ \tau\ !f\ u.$$

The left hand side of this equation becomes

$$fold\ \tau\ !f\ (u\ \mu\alpha.\sigma(\alpha)\ !in) = (u\ \mu\alpha.\sigma(\alpha)\ !in)\ \tau\ !f$$

and, since the right hand side is simply

$$u\ \tau\ !f,$$

the lemma follows from Lemma 1.37. \square

Theorem 2.24. *Consider an algebra $\Xi \mid -; - \vdash f: \sigma(\tau) \multimap \tau$ and a map of algebras $\Xi \mid -; - \vdash h: \mu\alpha.\sigma(\alpha) \multimap \tau$ from *in* to f . Then if we assume identity extension and extensionality, $h =_{\mu\alpha.\sigma(\alpha) \multimap \tau} fold\ \tau\ !f$.*

Proof. By Lemma 2.22 we have

$$h \circ (fold\ \mu\alpha.\sigma(\alpha)\ !in) =_{\mu\alpha.\sigma(\alpha) \multimap \tau} fold\ \tau\ !f.$$

Lemma 2.23 finishes the job. \square

We have shown that in defines an initial algebra.

In the logic, the initial algebras also satisfy an induction principle. We now show the following (relational) induction principle.

Theorem 2.25 (Induction). *Suppose $R: \text{AdmRel}(\mu\alpha.\sigma(\alpha), \mu\alpha.\sigma(\alpha))$ satisfies*

$$(in, in): \sigma[R] \multimap R.$$

Then

$$\forall x: \mu\alpha.\sigma(\alpha). R(x, x)$$

Remark 2.26. The induction principle speaks about relations since it is obtained as a consequence of binary parametricity. In case one also has unary parametricity available (for some notion of admissible propositions), applying the proof of Theorem 2.25 to unary parametricity will yield the well-known propositional induction principle: If ϕ is an *admissible* proposition on $\mu\alpha.\sigma(\alpha)$, then

$$(\forall x: \sigma(\mu\alpha.\sigma(\alpha)). \sigma[\phi](x) \supset \phi(in\ x)) \supset \forall x: \mu\alpha.\sigma(\alpha). \phi(x)$$

Proof of Theorem 2.25. By parametricity, for any $x: \mu\alpha.\sigma(\alpha)$,

$$x(\forall\alpha, \beta, R: \text{AdmRel}(\alpha, \beta). (\sigma[R] \multimap R) \rightarrow R)x$$

The assumption states that $(in, in): \sigma[R] \multimap R$ and so by Lemma 1.33

$$(!in, !in): !(\sigma[R] \multimap R).$$

Thus

$$R(x\ \mu\alpha.\sigma(\alpha)\ !in, x\ \mu\alpha.\sigma(\alpha)\ !in).$$

Finally, Lemma 2.23 tells us that $x\ \mu\alpha.\sigma(\alpha)\ !in = x$, which proves the theorem. \square

2.10. Final Coalgebras. As in section 2.9 we will assume that $\alpha \vdash \sigma(\alpha)$: **Type** is a type in which α occurs only positively, and this time we construct final coalgebras for the induced functor.

Define

$$\nu\alpha.\sigma(\alpha) = \coprod \alpha. !(\alpha \multimap \sigma(\alpha)) \otimes \alpha = \prod \beta. (\prod \alpha. (!(\alpha \multimap \sigma(\alpha)) \otimes \alpha \multimap \beta)) \multimap \beta$$

with combinators

$$\begin{aligned} \text{unfold}: \prod \alpha. (\alpha \multimap \sigma(\alpha)) &\rightarrow \alpha \multimap \nu\alpha.\sigma(\alpha), \\ \text{out}: \nu\alpha.\sigma(\alpha) &\multimap \sigma(\nu\alpha.\sigma(\alpha)) \end{aligned}$$

defined by

$$\begin{aligned} \text{unfold} &= \Lambda\alpha. \lambda^\circ f: !(\alpha \multimap \sigma(\alpha)). \lambda^\circ x: \alpha. \text{pack } \alpha (f \otimes x) \\ \text{out} &= \lambda^\circ x: \nu\alpha.\sigma(\alpha). x\ \sigma(\nu\alpha.\sigma(\alpha))\ r, \end{aligned}$$

where

$$\begin{aligned} r &: \prod \alpha. !(\alpha \multimap \sigma(\alpha)) \otimes \alpha \multimap \sigma(\nu\alpha.\sigma(\alpha)) \\ r &= \Lambda\alpha. \lambda^\circ y: !(\alpha \multimap \sigma(\alpha)) \otimes \alpha. \text{let } w \otimes z \text{ be } y \text{ in } \sigma(\text{unfold } \alpha\ w)(\text{let } !f \text{ be } w \text{ in } f\ z). \end{aligned}$$

Lemma 2.27. *For any coalgebra $f: \tau \multimap \sigma(\tau)$, the map $\text{unfold } \tau\ !f$ is a map of coalgebras from f to out .*

Proof. We need to prove that the following diagram commutes

$$\begin{array}{ccc}
 \tau & \xrightarrow{f} & \sigma(\tau) \\
 \text{unfold } \tau ! f \downarrow & & \downarrow \sigma(\text{unfold } \tau ! f) \\
 \nu\alpha. \sigma(\alpha) & \xrightarrow{\text{out}} & \sigma(\nu\alpha. \sigma(\alpha)).
 \end{array}$$

But this is done by a simple computation

$$\begin{aligned}
 \text{out}(\text{unfold } \tau ! f \ x) &= \text{out}(\text{pack } \tau (!f) \otimes x) \\
 &= \text{pack } \tau (!f) \otimes x \ \sigma(\nu\alpha. \sigma(\alpha)) \ r \\
 &= r \ \tau \ ((!f) \otimes x) \\
 &= \sigma(\text{unfold } \tau (!f)) (f \ x).
 \end{aligned}$$

□

Lemma 2.27 shows that *out* is a weakly final coalgebra for the functor induced by σ on $\mathbf{LinType}_\Xi$ for each Ξ . Notice that parametricity was not needed here.

Lemma 2.28. *Suppose $h: \tau \multimap \tau'$ is a map of coalgebras from $f: \tau \multimap \sigma(\tau)$ to $f': \tau' \multimap \sigma(\tau')$. If we assume identity extension, then the diagram*

$$\begin{array}{ccc}
 \tau & \xrightarrow{\text{unfold } \tau ! f} & \nu\alpha. \sigma(\alpha) \\
 h \downarrow & \nearrow \text{unfold } \tau' ! f' & \\
 \tau' & &
 \end{array}$$

commutes internally.

Proof. Using the Graph Lemma, the notion of h being a map of coalgebras can be expressed as

$$f(\langle h \rangle \multimap \sigma[\langle h \rangle])f'.$$

Now, by parametricity of *unfold*,

$$\text{unfold } \tau ! f(\langle h \rangle \multimap \text{eq}_{\nu\alpha. \sigma(\alpha)}) \text{unfold } \tau' ! f',$$

which is exactly what we wanted to prove. □

Lemma 2.29. *Given linear contexts C and C' , suppose*

$$\forall x: \sigma. \forall y: \tau. C[x \otimes y] =_\omega C'[x \otimes y].$$

then

$$\forall z: \sigma \otimes \tau. \text{let } x \otimes y \text{ be } z \text{ in } C[x \otimes y] =_\omega \text{let } x \otimes y \text{ be } z \text{ in } C'[x \otimes y]$$

Proof. Consider

$$f = \lambda^\circ x: \sigma. \lambda^\circ y: \tau. C[x \otimes y] \quad f' = \lambda^\circ x: \sigma. \lambda^\circ y: \tau. C'[x \otimes y]$$

then

$$f(\text{eq}_\sigma \multimap \text{eq}_\tau \multimap \text{eq}_\omega) f'.$$

If $z: \sigma \otimes \tau$ then by identity extension $\text{eq}_\sigma \otimes \text{eq}_\tau(z, z)$. By definition of $\text{eq}_\sigma \otimes \text{eq}_\tau$ we have

$$\text{let } x \otimes x' \text{ be } z \text{ in } fxx' =_\omega \text{let } x \otimes x' \text{ be } z \text{ in } f'xx'$$

which proves the lemma. □

Lemma 2.30. *Using extensionality and identity extension,*

$$\text{unfold } \nu\alpha. \sigma(\alpha) !out$$

is internally equal to the identity on $\nu\alpha. \sigma(\alpha)$.

Proof. Set $h = \text{unfold } \nu\alpha. \sigma(\alpha) !out$ in the following.

By Lemma 2.27 h is a map of coalgebras from out to out , so by Lemma 2.28, $h = h^2$. Intuitively, all we need to prove now is that h is “surjective”.

Consider any $g : \prod \alpha. (!(\alpha \multimap \sigma(\alpha)) \otimes \alpha \multimap \beta)$. For any coalgebra map $k : \alpha \multimap \alpha'$ from $f : \alpha \multimap \sigma(\alpha)$ to $f' : \alpha' \multimap \sigma(\alpha')$, we must have, by Lemmas 2.16, 1.33, and 1.32,

$$(!f \otimes x)(!(\langle k \rangle \multimap \sigma[\langle k \rangle]) \otimes \langle k \rangle)(!f' \otimes kx),$$

so by identity extension and parametricity of g ,

$$\forall x : \alpha. g \alpha (!f) \otimes x =_{\beta} g \alpha' (!f') \otimes k(x).$$

Using this on the coalgebra map $\text{unfold } \alpha !f$ from f to out we obtain

$$\forall x : \alpha. g \alpha (!f) \otimes x =_{\beta} g \nu\alpha. \sigma(\alpha) (!out) \otimes \text{unfold } \alpha !f x.$$

By Lemma 1.37 this implies that

$$\forall f : !(\alpha \multimap \sigma(\alpha)), x : \alpha. g \alpha f \otimes x =_{\beta} g \nu\alpha. \sigma(\alpha) (!out) \otimes \text{unfold } \alpha f x,$$

which implies

$$\forall z : !(\alpha \multimap \sigma(\alpha)) \otimes \alpha. g \alpha z =_{\beta} g \nu\alpha. \sigma(\alpha) (\text{let } f \otimes x \text{ be } z \text{ in } (!out) \otimes \text{unfold } \alpha f x)$$

using Lemma 2.29.

In other words, if we define

$$k : \prod \alpha. (!(\alpha \multimap \sigma(\alpha)) \otimes \alpha \multimap \tau),$$

where $\tau = !(\nu\alpha. \sigma(\alpha) \multimap \sigma(\nu\alpha. \sigma(\alpha))) \otimes \nu\alpha. \sigma(\alpha)$, to be

$$k = \Lambda\alpha. \lambda^{\circ} y : !(\alpha \multimap \sigma(\alpha)) \otimes \alpha. \text{let } f \otimes x \text{ be } y \text{ in } (!out) \otimes \text{unfold } \alpha f x,$$

then

$$\forall \alpha. g \alpha =_{!(\alpha \multimap \sigma(\alpha)) \otimes \alpha \multimap \beta} (g \nu\alpha. \sigma(\alpha)) \circ (k \alpha). \quad (2.2)$$

Now, suppose we are given $\alpha, \alpha', R : \text{Rel}(\alpha, \alpha')$ and terms f, f' such that

$$f(! (R \multimap \sigma[R]) \otimes R) f'.$$

Then, by (2.2) and parametricity of g

$$g \alpha f =_{\beta} g \alpha' f' =_{\beta} (g \nu\alpha. \sigma(\alpha))(k \alpha' f'),$$

from which we conclude

$$g(\forall(\alpha, \beta, R : \text{Rel}(\alpha, \beta)). (! (R \multimap \sigma[R]) \otimes R \multimap (g \nu\alpha. \sigma(\alpha))^{op}))k.$$

(Here we use S^{op} for the inverse relation of S .) Using parametricity, this implies that, for any $x : \nu\alpha. \sigma(\alpha)$, we have

$$x \beta g =_{\beta} g \nu\alpha. \sigma(\alpha) (x \tau k).$$

Thus, since g was arbitrary, we may apply the above to $g = k$ and get

$$x \tau k =_{\tau} k \nu\alpha. \sigma(\alpha) (x \tau k) = \text{let } f \otimes z \text{ be } (x \tau k) \text{ in } (!out) \otimes \text{unfold } \alpha f z.$$

If we write

$$l = \lambda x : \nu\alpha. \sigma(\alpha). \text{let } f \otimes z \text{ be } (x \tau k) \text{ in } \text{unfold } \alpha f z,$$

then, since k is a closed term, so is l , and from the above calculations we conclude that we have

$$\forall \beta. \forall g : \prod \alpha. !(\alpha \multimap \sigma(\alpha)) \otimes \alpha \multimap \beta. x \beta g =_{\beta} g \nu \alpha. \sigma(\alpha) (!out) \otimes (l x).$$

Now, finally,

$$\begin{aligned} h(l x) &= \text{unfold } \nu \alpha. \sigma(\alpha) !out (l x) \\ &= \text{pack } \nu \alpha. \sigma(\alpha) !out \otimes (l x) \\ &= \Lambda \beta. \lambda g : \prod \alpha. !(\alpha \multimap \sigma(\alpha)) \otimes \alpha \multimap \beta. g \nu \alpha. \sigma(\alpha) !out \otimes (l x) \\ &=_{\nu \alpha. \sigma(\alpha)} \Lambda \beta. \lambda g : \prod \alpha. !(\alpha \multimap \sigma(\alpha)) \otimes \alpha \multimap \beta. x \beta g \\ &= x, \end{aligned}$$

where we have used extensionality. Thus l is a right inverse to h , and we conclude

$$h x =_{\nu \alpha. \sigma(\alpha)} h^2(l x) =_{\nu \alpha. \sigma(\alpha)} h(l x) =_{\nu \alpha. \sigma(\alpha)} x.$$

□

Theorem 2.31. *Suppose $\Xi \mid -; - \vdash f : \tau \multimap \sigma(\tau)$ is a coalgebra and $\Xi \mid -; - \vdash h : \tau \multimap \mu \alpha. \sigma(\alpha)$ is a map of algebras from f to out . Then if we assume identity extension and extensionality $h =_{\tau \multimap \mu \alpha. \sigma(\alpha)} \text{unfold } \alpha !f$.*

Proof. Consider a map of coalgebras into out :

$$\begin{array}{ccc} \tau & \xrightarrow{f} & \sigma(\tau) \\ \downarrow h & & \downarrow \sigma(h) \\ \nu \alpha. \sigma(\alpha) & \xrightarrow{out} & \sigma(\nu \alpha. \sigma(\alpha)). \end{array}$$

By Lemmas 2.28 and 2.30,

$$\text{unfold } \tau !f =_{\tau \multimap \nu \alpha. \sigma(\alpha)} (\text{unfold } \nu \alpha. \sigma(\alpha) !out) \circ g =_{\tau \multimap \nu \alpha. \sigma(\alpha)} g.$$

□

Theorem 2.31 shows that out is a final coalgebra for the endofunctor on **LinType** $_{\Xi}$ induced by σ for each Ξ .

We now show how the final coalgebras satisfy a coinduction principle.

Theorem 2.32 (Coinduction). *Suppose that $R : \text{AdmRel}(\nu \alpha. \sigma(\alpha), \nu \alpha. \sigma(\alpha))$ is such that*

$$(out, out) : R \multimap \sigma[R].$$

We then have that

$$\forall x, y : \nu \alpha. \sigma(\alpha). R(x, y) \supset x =_{\nu \alpha. \sigma(\alpha)} y.$$

Proof. Suppose $R : \text{AdmRel}(\nu \alpha. \sigma(\alpha), \nu \alpha. \sigma(\alpha))$ satisfies $(out, out) : R \multimap \sigma[R]$ and $R(x, y)$. By parametricity of

$$\text{pack} : \prod \alpha. !(\alpha \multimap \sigma(\alpha)) \otimes \alpha \multimap \nu \alpha. \sigma(\alpha)$$

we have

$$\text{pack } \nu \alpha. \sigma(\alpha) !out \otimes x =_{\nu \alpha. \sigma(\alpha)} \text{pack } \nu \alpha. \sigma(\alpha) !out \otimes y$$

and by 2.30,

$$\begin{aligned} \text{pack } \nu \alpha. \sigma(\alpha) !out \otimes x &=_{\nu \alpha. \sigma(\alpha)} x \\ \text{pack } \nu \alpha. \sigma(\alpha) !out \otimes y &=_{\nu \alpha. \sigma(\alpha)} y \end{aligned}$$

which proves the theorem. □

The next theorem is an interesting generalization of Theorem 2.32, stating that the assumption of admissibility in the coinduction principle is unnecessary. A similar result was proved by Pitts in the setting of coinductive types in the category of domains [Pit95]. To state this theorem we need again to use the general hypothesis of this section that σ is an inductively defined type, since in this case we can define $\sigma[R]$ for general (not just admissible) relations inductively over the structure of σ using the constructions of Section 1.2.2. Recall that for more general types σ the construction $\sigma[R]$ is defined as in Figure 4 for admissible relations R only.

Theorem 2.33 (General coinduction principle). *Suppose $R: \text{Rel}(\nu\alpha.\sigma(\alpha), \nu\alpha.\sigma(\alpha))$ is a relation such that $(out, out): R \multimap \sigma[R]$, then*

$$\forall x, y: \nu\alpha.\sigma(\alpha). R(x, y) \supset x =_{\nu\alpha.\sigma(\alpha)} y$$

Proof. Suppose $R: \text{Rel}(\nu\alpha.\sigma(\alpha), \nu\alpha.\sigma(\alpha))$ is any relation satisfying $(out, out): R \multimap \sigma[R]$. The idea of the proof is to use Theorem 2.32 on the admissible relation $\Phi(R)$. Since by Lemma 1.35 Φ is a functor,

$$(out, out): \Phi(R) \multimap \Phi(\sigma[R]),$$

and since $\sigma[\Phi(R)]$ is an admissible relation containing $\sigma[R]$, and $\Phi(\sigma[R])$ is the smallest such, we have $\Phi(\sigma[R]) \subset \sigma[\Phi(R)]$ and so

$$(out, out): \Phi(R) \multimap \sigma[\Phi(R)].$$

Now, the coinduction principle for admissible relations gives us

$$\forall x, y: \nu\alpha.\sigma(\alpha). \Phi(R)(x, y) \supset x =_{\nu\alpha.\sigma(\alpha)} y$$

and so the theorem follows from $R \subset \Phi(R)$. □

2.11. Recursive type equations. In this section we consider inductively constructed types $\alpha \vdash \sigma(\alpha)$ and construct closed types $\text{rec } \alpha.\sigma(\alpha)$ such that $\sigma(\text{rec } \alpha.\sigma(\alpha)) \cong \text{rec } \alpha.\sigma(\alpha)$. In Sections 2.9 and 2.10 we solved the problem in the special case of α occurring only positively in σ , by finding initial algebras and final coalgebras for the functor induced by σ .

This section details the sketch of [Plo93], but the theory is due to Freyd [Fre90b, Fre90a, Fre91]. In short, the main observation is that because of the presence of fixed points, the initial algebras and final coalgebras of Sections 2.9, 2.10 coincide (Theorem 2.39 below). This phenomenon is called algebraic compactness, and was studied by Freyd in *loc. cit.*. Using Freyd's techniques we find solutions to recursive type equations as advertised, and show that they satisfy a universal property called the initial dialgebra property. Moreover, we generalize the induction and coinduction properties of Theorems 2.25, 2.33 to a combined induction/coinduction property for recursive types. In Section 2.13 we treat the case of recursive type equations with parameters.

Before we start, observe that we may split the occurrences of α in σ into positive and negative occurrences. So our standard assumption in this section is that we are given a type $\alpha, \beta \vdash \sigma(\alpha, \beta)$, in which α occurs only negatively and β only positively, and we look for a type $\text{rec } \alpha.\sigma(\alpha, \alpha)$ isomorphic to $\sigma(\text{rec } \alpha.\sigma(\alpha, \alpha), \text{rec } \alpha.\sigma(\alpha, \alpha))$. In this notation, $\text{rec } \alpha.\sigma(\alpha, \alpha)$ binds α in σ .

2.11.1. *Parametrized initial algebras.* Set $\omega(\alpha) = \mu\beta. \sigma(\alpha, \beta) = \prod \beta. (\sigma(\alpha, \beta) \multimap \beta) \multimap \beta$. Now, ω induces a contravariant functor from types to types.

Lemma 2.34. *Assuming identity extension and extensionality, for $f: \alpha' \multimap \alpha$, up to internal equality $\omega(f): \omega(\alpha) \multimap \omega(\alpha')$ is the unique h such that*

$$\begin{array}{ccc} \sigma(\alpha, \omega(\alpha)) & \xrightarrow{\text{in}} & \omega(\alpha) \\ \sigma(\text{id}, h) \downarrow & & \downarrow h \\ \sigma(\alpha, \omega(\alpha')) & & \\ \sigma(f, \text{id}) \downarrow & & \\ \sigma(\alpha', \omega(\alpha')) & \xrightarrow{\text{in}} & \omega(\alpha') \end{array}$$

commutes internally.

Proof. One may define in as a polymorphic term

$$\text{in}: \prod \alpha. \sigma(\alpha, \omega(\alpha)) \multimap \omega(\alpha)$$

by

$$\text{in} = \Lambda \alpha. \lambda^\circ z: \sigma(\alpha, \omega(\alpha)). \Lambda \beta. \lambda f: \sigma(\alpha, \beta) \multimap \beta. f(\sigma(\lambda x: \alpha. x, \text{fold } \beta ! f) z).$$

By parametricity we have

$$\text{in } \alpha'(\sigma(\langle f \rangle, \omega(\langle f \rangle))) \multimap \omega(\langle f \rangle) \text{in } \alpha,$$

which, by the Graph Lemma (Lemma 2.16), means that

$$\text{in } \alpha'(\langle \sigma(f, \omega(f)) \rangle^{\text{op}} \multimap \langle \omega(f) \rangle^{\text{op}}) \text{in } \alpha,$$

which in turn amounts to internal commutativity of the diagram of the lemma.

Uniqueness is by initiality of in (in $\mathbf{LinType}_\alpha$, proved as before) used on the diagram

$$\begin{array}{ccc} \sigma(\alpha, \omega(\alpha)) & \xrightarrow{\text{in}} & \omega(\alpha) \\ \sigma(\text{id}, h) \downarrow & & \downarrow h \\ \sigma(\alpha, \omega(\alpha')) & \xrightarrow{\sigma(f, \text{id})} \sigma(\alpha', \omega(\alpha')) & \xrightarrow{\text{in}} \omega(\alpha'). \end{array}$$

□

2.11.2. *Dialgebras.*

Definition 2.35. A dialgebra for σ is a quadruple (τ, τ', f, f') such that τ and τ' are types, and $f: \sigma(\tau', \tau) \multimap \tau$ and $f': \tau' \multimap \sigma(\tau, \tau')$ are morphisms. A morphism of dialgebras from dialgebra $(\tau_0, \tau'_0, f_0, f'_0)$ to $(\tau_1, \tau'_1, f_1, f'_1)$ is a pair of morphisms $h: \tau_0 \multimap \tau_1$, $h': \tau'_1 \multimap \tau'_0$, such that

$$\begin{array}{ccc} \sigma(\tau'_0, \tau_0) & \xrightarrow{f_0} & \tau_0 \\ \sigma(h', h) \downarrow & & \downarrow h \\ \sigma(\tau'_1, \tau_1) & \xrightarrow{f_1} & \tau_1 \end{array} \quad \begin{array}{ccc} \tau'_1 & \xrightarrow{f'_1} & \sigma(\tau_1, \tau'_1) \\ h' \downarrow & & \downarrow \sigma(h, h') \\ \tau'_0 & \xrightarrow{f'_0} & \sigma(\tau_0, \tau'_0). \end{array}$$

Lemma 2.36. *If (h, h') is a map of dialgebras and h, h' are isomorphisms, then (h, h') is an isomorphism of dialgebras.*

Proof. The only thing to prove here is that $(h^{-1}, (h')^{-1})$ is in fact a map of dialgebras, which is trivial. \square

Remark 2.37. If we for the type $\alpha, \beta \vdash \sigma : \mathbf{Type}$ consider for each Ξ the endofunctor

$$\langle \sigma^{\text{op}}, \sigma \rangle : \mathbf{LinType}^{\text{op}}_{\Xi} \times \mathbf{LinType}_{\Xi} \rightarrow \mathbf{LinType}^{\text{op}}_{\Xi} \times \mathbf{LinType}_{\Xi}$$

defined by $(\alpha, \beta) \mapsto (\sigma(\beta, \alpha), \sigma(\alpha, \beta))$, then dialgebras for σ are exactly the algebras for $\langle \sigma^{\text{op}}, \sigma \rangle$, maps of dialgebras are maps of algebras for $\langle \sigma^{\text{op}}, \sigma \rangle$ and initial dialgebras correspond to initial algebras. Dialgebras as considered here are a special case of what Hagino calls F, G -dialgebras in his thesis [Hag87], for F being $\langle \sigma^{\text{op}}, \sigma \rangle$ and G being the identity functor.

Theorem 2.38. *Assuming identity extension and extensionality, initial dialgebras exist for all functors induced by types $\sigma(\alpha, \beta)$, up to internal equality.*

Proof. In this proof, commutativity of diagrams will mean commutativity up to internal equality.

Set $\omega(\alpha) = \mu\beta. \sigma(\alpha, \beta)$. Then, ω defines a contravariant functor. Define

$$\tau' = \nu\alpha. \sigma(\omega(\alpha), \alpha), \quad \tau = \omega(\tau') = \mu\beta. \sigma(\tau', \beta).$$

Since τ' is defined as the final coalgebra for a functor, we have a morphism

$$\text{out} : \tau' \multimap \sigma(\omega(\tau'), \tau') = \sigma(\tau, \tau'),$$

and since τ is defined to be an initial algebra, we get a morphism

$$\text{in} : \sigma(\tau', \tau) \multimap \tau.$$

We will show that $(\tau, \tau', \text{in}, \text{out})$ is an initial dialgebra.

Suppose we are given a dialgebra (τ_0, τ'_0, g, g') . Since in is an initial algebra, there exists a unique map a , such that

$$\begin{array}{ccc} \sigma(\tau'_0, \omega(\tau'_0)) & \xrightarrow{\text{in}} & \omega(\tau'_0) \\ \sigma(\text{id}, a) \downarrow & & \downarrow a \\ \sigma(\tau'_0, \tau_0) & \xrightarrow{g} & \tau_0 \end{array}$$

and thus, since out is a final coalgebra, we find a map h' making the diagram

$$\begin{array}{ccc} \tau'_0 & \xrightarrow{g'} & \sigma(\tau_0, \tau'_0) \xrightarrow{\sigma(a, \text{id})} \sigma(\omega(\tau'_0), \tau'_0) \\ h' \downarrow & & \downarrow \sigma(\omega(h'), h') \\ \tau' & \xrightarrow{\text{out}} & \sigma(\omega(\tau'), \tau') \end{array} \quad (2.3)$$

commute. Set $h = a \circ \omega(h')$. We claim that (h, h') defines a map of dialgebras. The second diagram of Definition 2.35 is simply (2.3). The first diagram of 2.35 follows from

the commutativity of the composite diagram

$$\begin{array}{ccc}
 \sigma(\tau', \omega(\tau')) & \xrightarrow{\text{in}} & \omega(\tau') \\
 \sigma(h', \omega(h')) \downarrow & & \downarrow \omega(h') \\
 \sigma(\tau'_0, \omega(\tau'_0)) & \xrightarrow{\text{in}} & \omega(\tau'_0) \\
 \sigma(id, a) \downarrow & & \downarrow a \\
 \sigma(\tau'_0, \tau_0) & \xrightarrow{g} & \tau_0,
 \end{array} \tag{2.4}$$

where the top diagram commutes by Lemma 2.34.

Finally, we will prove that (h, h') is the unique dialgebra morphism. Suppose we are given a map of dialgebras (k, k') from (τ, τ', in, out) to (τ_0, τ'_0, g, g') . By the first diagram of Definition 2.35, we have a commutative diagram

$$\begin{array}{ccc}
 \sigma(\tau', \tau) & \xrightarrow{\text{in}} & \tau \\
 \sigma(id, k) \downarrow & & \downarrow k \\
 \sigma(\tau', \tau_0) & \xrightarrow{\sigma(k', id)} \sigma(\tau'_0, \tau_0) \xrightarrow{g} & \tau_0.
 \end{array}$$

Since clearly (2.4) also commutes when k' is substituted for h' , by (strong) initiality of in , we conclude that $k =_{\tau \multimap \tau'} a \circ \omega(k')$. Finally, by the second diagram of Definition 2.35 we have commutativity of

$$\begin{array}{ccc}
 \tau'_0 & \xrightarrow{g'} \sigma(\tau_0, \tau'_0) \xrightarrow{\sigma(a, id)} \sigma(\omega(\tau'_0), \tau'_0) \\
 k' \downarrow & & \downarrow \sigma(\omega(k'), k') \\
 \tau' & \xrightarrow{\text{out}} \sigma(\omega(\tau'), \tau').
 \end{array}$$

So since out is a final coalgebra we conclude $k' =_{\tau'_0 \multimap \tau'} h'$. \square

2.11.3. Algebraic compactness. As advertised in the introduction to this section, the presence of fixed points makes initial algebras and final coalgebras coincide.

Theorem 2.39 (Algebraic compactness). *Assuming identity extension and extensionality, for all types $\alpha \vdash \sigma(\alpha)$ in which α occurs only positively, in^{-1} is internally a final coalgebra and out^{-1} is internally an initial algebra. Furthermore in^{-1} and out^{-1} can be written as terms of $PILL_Y$.*

Proof. By Theorems 2.24 and 2.31 in is an initial algebra, and out is a final coalgebra for σ . Consider

$$h = Y(\nu\alpha. \sigma(\alpha)) \multimap \mu\alpha. \sigma(\alpha) (\lambda h: \nu\alpha. \sigma(\alpha) \multimap \mu\alpha. \sigma(\alpha). in \circ \sigma(h) \circ out).$$

Since Y is a fixed-point operator, we know that

$$\begin{array}{ccc}
 \sigma(\nu\alpha. \sigma(\alpha)) & \xrightarrow{out} & \nu\alpha. \sigma(\alpha) \\
 \sigma(h) \downarrow & & \downarrow h \\
 \sigma(\mu\alpha. \sigma(\alpha)) & \xrightarrow{in} & \mu\alpha. \sigma(\alpha)
 \end{array}$$

commutes. Since in^{-1} is a coalgebra, we also have a map k going the other way, and since out is a final coalgebra, $k \circ h =_{\nu\alpha.\sigma(\alpha) \dashv \nu\alpha.\sigma(\alpha)} id_{\nu\alpha.\sigma(\alpha)}$. Since in is an initial algebra, we know that $h \circ k =_{\mu\alpha.\sigma(\alpha) \dashv \mu\alpha.\sigma(\alpha)} id_{\mu\alpha.\sigma(\alpha)}$. So $in^{-1} \cong out$ as coalgebras and $out^{-1} \cong in$ as algebras, internally. \square

Lemma 2.40. *Assume identity extension and extensionality. Let (τ, τ', in, out) be the initial dialgebra from the proof of Theorem 2.38. Then $(\tau', \tau, out^{-1}, in^{-1})$ is also an initial dialgebra internally.*

Proof. In this proof, commutativity of diagrams is up to internal equality.

Suppose we are given a dialgebra (τ_0, τ'_0, g, g') . We will show that there exists a unique morphism of dialgebras from $(\tau', \tau, out^{-1}, in^{-1})$ to (τ_0, τ'_0, g, g') .

By Theorem 2.39, for all types α , $in^{-1}: \omega(\alpha) \dashv \sigma(\alpha, \omega(\alpha))$ is a final coalgebra for the functor $\beta \mapsto \sigma(\alpha, \beta)$, and $out^{-1}: \sigma(\tau, \tau') \dashv \tau'$ is an initial algebra for the functor $\alpha \mapsto \sigma(\omega(\alpha), \alpha)$.

Let a be the unique map making the diagram

$$\begin{array}{ccc} \tau'_0 & \xrightarrow{g'} & \sigma(\tau_0, \tau'_0) \\ a \downarrow & & \downarrow \sigma(id, a) \\ \omega(\tau_0) & \xrightarrow{in^{-1}} & \sigma(\tau_0, \omega(\tau_0)) \end{array}$$

commute. Define h to be the unique map making

$$\begin{array}{ccc} \sigma(\tau, \tau') & \xrightarrow{out^{-1}} & \tau' \\ \sigma(\omega(h), h) \downarrow & & \downarrow h \\ \sigma(\omega(\tau_0), \tau_0) & \xrightarrow{\sigma(a, id)} & \sigma(\tau'_0, \tau_0) \xrightarrow{g} \tau_0 \end{array} \quad (2.5)$$

commute. We define h' to be $\omega(h) \circ a$ and prove that (h, h') is a map of dialgebras. The first diagram of Definition 2.35 is simply (2.5). Commutativity of the second diagram follows from commutativity of

$$\begin{array}{ccc} \tau'_0 & \xrightarrow{g'} & \sigma(\tau_0, \tau'_0) \\ a \downarrow & & \downarrow \sigma(id, a) \\ \omega(\tau_0) & \xrightarrow{in^{-1}} & \sigma(\tau_0, \omega(\tau_0)) \\ \omega(h) \downarrow & & \downarrow \sigma(h, \omega(h)) \\ \omega(\tau') & \xrightarrow{in^{-1}} & \sigma(\tau', \omega(\tau')), \end{array} \quad (2.6)$$

where commutativity of the last diagram follows from Lemma 2.34.

Finally, we will show that if (k, k') is another map of dialgebras from the dialgebra $(\tau', \tau, out^{-1}, in^{-1})$ to (τ_0, τ'_0, g, g') then $h =_{\tau' \dashv \tau_0} k$ and $h' =_{\tau'_0 \dashv \tau} k'$. By the second diagram of Definition 2.35 we know that

$$\begin{array}{ccc} \tau'_0 & \xrightarrow{g'} & \sigma(\tau_0, \tau'_0) \xrightarrow{\sigma(k, id)} \sigma(\tau', \tau'_0) \\ k' \downarrow & & \downarrow \sigma(id, k') \\ \tau & \xrightarrow{in^{-1}} & \sigma(\tau', \tau) \end{array} \quad (2.7)$$

commutes. Clearly, if we substitute k for h in (2.6), we obtain a diagram that commutes by Lemma 2.34. So, using the fact that in^{-1} is a final coalgebra on (2.7), we get $k' =_{\tau'_0 \multimap \sigma\tau} \omega(k) \circ a$.

The first diagram of Definition 2.35 implies that

$$\begin{array}{ccc} \sigma(\tau, \tau') & \xrightarrow{out^{-1}} & \tau' \\ \sigma(\omega(k), k) \downarrow & & \downarrow k \\ \sigma(\omega(\tau_0), \tau_0) & \xrightarrow{\sigma(a, id)} \sigma(\tau'_0, \tau_0) & \xrightarrow{g} \tau_0 \end{array}$$

commutes. Comparing this to (2.5) we obtain $h =_{\tau' \multimap \sigma\tau_0} k$, by initiality of out^{-1} . \square

Theorem 2.41. *Assuming identity extension and extensionality, for all types $\sigma(\alpha, \beta)$ where α occurs only negatively and β only positively, there exists a type $rec \alpha. \sigma(\alpha, \alpha)$ and an isomorphism*

$$i: \sigma(rec \alpha. \sigma(\alpha, \alpha), rec \alpha. \sigma(\alpha, \alpha)) \multimap rec \alpha. \sigma(\alpha, \alpha),$$

such that $(rec \alpha. \sigma(\alpha, \alpha), rec \alpha. \sigma(\alpha, \alpha), i, i^{-1})$ is an initial dialgebra up to internal equality.

Proof. As usual commutativity of diagrams will be up to internal equality.

We have a unique map of dialgebras

$$(h, h'): (\tau, \tau', in, out) \rightarrow (\tau', \tau, out^{-1}, in^{-1})$$

We claim that (h', h) is also a map of dialgebras from (τ, τ', in, out) to $(\tau', \tau, out^{-1}, in^{-1})$.

To prove this we need to prove commutativity of the diagrams

$$\begin{array}{ccc} \sigma(\tau', \tau) & \xrightarrow{in} & \tau \\ \sigma(h, h') \downarrow & & \downarrow h' \\ \sigma(\tau, \tau') & \xrightarrow{out^{-1}} & \tau' \end{array} \quad , \quad \begin{array}{ccc} \tau & \xrightarrow{in^{-1}} & \sigma(\tau', \tau) \\ h \downarrow & & \downarrow \sigma(h', h) \\ \tau' & \xrightarrow{out} & \sigma(\tau, \tau') \end{array}$$

but the fact that (h, h') is a map of dialgebras tells us exactly that

$$\begin{array}{ccc} \sigma(\tau', \tau) & \xrightarrow{in} & \tau \\ \sigma(h', h) \downarrow & & \downarrow h \\ \sigma(\tau, \tau') & \xrightarrow{out^{-1}} & \tau' \end{array} \quad \begin{array}{ccc} \tau & \xrightarrow{in^{-1}} & \sigma(\tau', \tau) \\ h' \downarrow & & \downarrow \sigma(h, h') \\ \tau' & \xrightarrow{out} & \sigma(\tau, \tau'), \end{array}$$

and these two diagram are the same as the above but in opposite order. Thus, by uniqueness of maps of dialgebras out of (τ, τ', in, out) , we get $h =_{\tau \multimap \sigma\tau'} h'$. Since (h, h) is a map between initial dialgebras, h is an isomorphism.

Now define $f: \sigma(\tau, \tau) \multimap \tau$ to be $in \circ \sigma(h^{-1}, id_\tau)$. Then clearly (id_τ, h^{-1}) is a morphism of dialgebras from (τ, τ, f, f^{-1}) to (τ, τ', in, out) , since the diagrams proving (id_τ, h^{-1}) to be a map of dialgebras are

$$\begin{array}{ccc} \sigma(\tau, \tau) & \xrightarrow{\sigma(h^{-1}, id)} \sigma(\tau', \tau) & \xrightarrow{in} \tau \\ \sigma(h^{-1}, id) \downarrow & \searrow f & \downarrow id \\ \sigma(\tau', \tau) & \xrightarrow{in} & \tau \end{array} \quad \begin{array}{ccc} \tau' & \xrightarrow{out} & \sigma(\tau, \tau') \\ h^{-1} \downarrow & & \downarrow \sigma(id, h^{-1}) \\ \tau & \xrightarrow{in^{-1}} \sigma(\tau', \tau) & \xrightarrow{\sigma(h, id)} \sigma(\tau, \tau) \\ & \searrow f^{-1} & \end{array}$$

Clearly the first diagram commutes, and the second diagram is just part of the definition of (h, h) being a map of dialgebras. Thus (id_τ, h^{-1}) defines an isomorphism of dialgebras from (τ, τ, f, f^{-1}) to (τ, τ', in, out) , as desired. \square

Notice that the closed terms $rec \alpha. \sigma(\alpha, \alpha) \multimap \sigma(rec \alpha. \sigma(\alpha, \alpha), rec \alpha. \sigma(\alpha, \alpha))$ and

$$\sigma(rec \alpha. \sigma(\alpha, \alpha), rec \alpha. \sigma(\alpha, \alpha)) \multimap rec \alpha. \sigma(\alpha, \alpha)$$

always exist, independent of the assumption of parametricity. Parametricity implies that they are each others inverses.

2.12. A mixed induction/coinduction principle. Here we prove the following reasoning principle for the recursive type $rec \alpha. \sigma(\alpha, \alpha)$. This principle is the same as the one obtained by Pitts for recursive types in the category domains [Pit95, Cor 4.10]. Again, as noted before Theorem 2.33 we must assume that σ is an inductively defined type to make sense of the relational interpretation of σ at general non-admissible relations.

Theorem 2.42. *Suppose $\alpha, \beta \vdash \sigma(\alpha, \beta)$ is an inductively defined type in which α occurs only positively and β only negatively. Suppose further*

$$\begin{aligned} R^- &: \text{Rel}(rec \alpha. \sigma(\alpha, \alpha), rec \alpha. \sigma(\alpha, \alpha)) \text{ and} \\ R^+ &: \text{AdmRel}(rec \alpha. \sigma(\alpha, \alpha), rec \alpha. \sigma(\alpha, \alpha)) \end{aligned}$$

are relations. Then the following principle holds

$$\frac{(i^{-1}, i^{-1}): R^- \multimap \sigma(R^+, R^-) \quad (i, i): \sigma(R^-, R^+) \multimap R^+}{R^- \subset eq_{rec \alpha. \sigma(\alpha, \alpha)} \subset R^+}$$

where i denotes the isomorphism

$$\sigma(rec \alpha. \sigma(\alpha, \alpha), rec \alpha. \sigma(\alpha, \alpha)) \multimap rec \alpha. \sigma(\alpha, \alpha).$$

Proof. We first prove the rule in the case of both relations being admissible. The proof in this case is a surprisingly simple consequence of parametricity.

The proof of Theorem 2.41 is constructive in the sense that there is a construction of the maps h, h' constituting the unique dialgebra map out of the initial dialgebra from the given types ω, ω' and terms t, t' . In fact, from the proof we can derive terms

$$\begin{aligned} k &: \prod \omega, \omega'. (\sigma(\omega', \omega) \multimap \omega) \multimap (\omega' \multimap \sigma(\omega, \omega')) \multimap rec \alpha. \sigma(\alpha, \alpha) \multimap \omega \\ k' &: \prod \omega, \omega'. (\sigma(\omega', \omega) \multimap \omega) \multimap (\omega' \multimap \sigma(\omega, \omega')) \multimap \omega' \multimap rec \alpha. \sigma(\alpha, \alpha) \end{aligned}$$

such that the maps h, h' can be obtained as

$$\begin{aligned} h &= k \omega \omega' t t' \\ h' &= k' \omega \omega' t t' \end{aligned}$$

The exact constructions of k, k' are not of interest us right now — what matters to us is that we can use the assumption of parametricity on them. We consider the case $\omega = \omega' = rec \alpha. \sigma(\alpha, \alpha)$ and $t = i$ and $t' = i^{-1}$. In this case of course $h = h' = id$. If we use parametricity of k' by substituting the relation R^- for the type ω' and R^+ for ω then we get since

$$id = k rec \alpha. \sigma(\alpha, \alpha) rec \alpha. \sigma(\alpha, \alpha) i i^{-1}$$

$(id, id): R^- \multimap eq_{rec \alpha. \sigma(\alpha, \alpha)}$. Likewise, using parametricity of k we get

$$(id, id): eq_{rec \alpha. \sigma(\alpha, \alpha)} \multimap R^+$$

which proves the theorem in the case of R^- being admissible.

For the general case, we just need a simple application of the closure operator of Lemma 1.35. So assume again

$$\begin{aligned} (i^{-1}, i^{-1}) &: R^- \multimap \sigma(R^+, R^-), \\ (i, i) &: \sigma(R^-, R^+) \multimap R^+, \end{aligned}$$

and R^+ is admissible, but R^- may not be. The idea is to use the case above on $\Phi(R^-)$ and R^+ which are both admissible, but we need to check that the hypothesis still holds for this case. First, by Φ being a functor

$$(i^{-1}, i^{-1}) : \Phi(R^-) \multimap \Phi(\sigma(R^+, R^-)).$$

But, since $\sigma(R^+, \Phi(R^-))$ is an admissible relation containing $\sigma(R^+, R^-)$,

$$\Phi(\sigma(R^+, R^-)) \subset \sigma(R^+, \Phi(R^-))$$

and so

$$(i^{-1}, i^{-1}) : \Phi(R^-) \multimap \sigma(R^+, \Phi(R^-)). \quad (2.8)$$

Since $\sigma(\Phi(R^-), R^+) \subset \sigma(R^-, R^+)$ we also have

$$(i, i) : \sigma(\Phi(R^-), R^+) \multimap R^+. \quad (2.9)$$

Using the case of admissible relation proved above on (2.8) and (2.9), we get

$$\Phi(R^-) \subset eq_{\text{rec } \alpha. \sigma(\alpha, \alpha)} \subset R^+$$

which together with $R^- \subset \Phi(R^-)$ proves the theorem in the general case. \square

2.13. Recursive type equations with parameters. We now consider recursive type equations with parameters, i.e., we consider types $\vec{\alpha}, \alpha \vdash \sigma(\vec{\alpha}, \alpha)$ and look for types $\vec{\alpha} \vdash \text{rec } \alpha. \sigma(\vec{\alpha}, \alpha)$ satisfying $\sigma(\vec{\alpha}, \text{rec } \alpha. \sigma(\vec{\alpha}, \alpha)) \cong \text{rec } \alpha. \sigma(\vec{\alpha}, \alpha)$. As before, we need to split occurrences of the variable α into positive and negative occurrences, and since we would like to be able to construct nested recursive types, we need to keep track of positive and negative occurrences of the variables $\vec{\alpha}$ in the solution $\text{rec } \alpha. \sigma(\vec{\alpha}, \alpha)$ as well. So we will suppose that we are given a type $\vec{\alpha}, \vec{\beta}, \alpha, \beta \vdash \sigma(\vec{\alpha}, \vec{\beta}, \alpha, \beta)$ in which the variables $\vec{\alpha}, \alpha$ occur only negatively and the variables $\vec{\beta}, \beta$ only positively.

Of course, the proof proceeds as in the case without parameters. However, one must take care to obtain the right occurrences of parameters, and so we sketch the proof here.

Lemma 2.43. *Suppose $\vec{\alpha}, \vec{\beta}, \alpha, \beta \vdash \sigma(\vec{\alpha}, \vec{\beta}, \alpha, \beta)$ is a type in which the variables $\vec{\alpha}, \alpha$ occur only negatively and the variables $\vec{\beta}, \beta$ only positively. There exists types $\vec{\alpha}, \vec{\beta} \vdash \tau(\vec{\alpha}, \vec{\beta})$ in which $\vec{\alpha}$ occurs only negatively and $\vec{\beta}$ only positively and $\vec{\alpha}, \vec{\beta} \vdash \tau'(\vec{\alpha}, \vec{\beta})$ in which $\vec{\alpha}$ occurs only positively and $\vec{\beta}$ only negatively and terms*

$$\begin{aligned} \text{in} &: \sigma(\vec{\alpha}, \vec{\beta}, \tau'(\vec{\alpha}, \vec{\beta}), \tau(\vec{\alpha}, \vec{\beta})) \multimap \tau(\vec{\alpha}, \vec{\beta}) \\ \text{out} &: \tau'(\vec{\alpha}, \vec{\beta}) \multimap \sigma(\vec{\beta}, \vec{\alpha}, \tau(\vec{\alpha}, \vec{\beta}), \tau'(\vec{\alpha}, \vec{\beta})) \end{aligned}$$

such that for any pair of types $\vec{\alpha}, \vec{\beta} \vdash \omega, \omega'$, and terms

$$\begin{aligned} g &: \sigma(\vec{\alpha}, \vec{\beta}, \omega', \omega) \multimap \omega \\ g' &: \omega' \multimap \sigma(\vec{\beta}, \vec{\alpha}, \omega, \omega') \end{aligned}$$

there exists unique h, h' making

$$\begin{array}{ccc}
 \sigma(\vec{\alpha}, \vec{\beta}, \tau'(\vec{\alpha}, \vec{\beta}), \tau(\vec{\alpha}, \vec{\beta})) & \xrightarrow{\text{in}} & \tau(\vec{\alpha}, \vec{\beta}) \\
 \sigma(\vec{\alpha}, \vec{\beta}, h', h) \downarrow & & \downarrow h \\
 \sigma(\vec{\alpha}, \vec{\beta}, \omega', \omega) & \xrightarrow{g} & \omega
 \end{array}
 \quad
 \begin{array}{ccc}
 \omega' & \xrightarrow{g'} & \sigma(\vec{\beta}, \vec{\alpha}, \omega, \omega') \\
 h' \downarrow & & \downarrow \sigma(\vec{\beta}, \vec{\alpha}, h, h') \\
 \tau'(\vec{\alpha}, \vec{\beta}) & \xrightarrow{\text{out}} & \sigma(\vec{\beta}, \vec{\alpha}, \tau(\vec{\alpha}, \vec{\beta}), \tau'(\vec{\alpha}, \vec{\beta}))
 \end{array}$$

commute up to internal equality.

Proof. Define

$$\begin{aligned}
 \omega(\vec{\alpha}, \vec{\beta}, \alpha) &= \mu\beta. \sigma(\vec{\alpha}, \vec{\beta}, \alpha, \beta) \\
 \tau'(\vec{\alpha}, \vec{\beta}) &= \nu\alpha. \sigma(\vec{\beta}, \vec{\alpha}, \omega(\vec{\alpha}, \vec{\beta}, \alpha), \alpha) \\
 \tau(\vec{\alpha}, \vec{\beta}) &= \omega(\vec{\alpha}, \vec{\beta}, \tau'(\vec{\alpha}, \vec{\beta}))
 \end{aligned}$$

Notice that we have swapped the occurrences of $\vec{\alpha}, \vec{\beta}$ in σ in the definition of τ' , making all occurrences of $\vec{\alpha}$ in τ' positive and all occurrences of $\vec{\beta}$ in τ' negative. The rest of the proof proceeds exactly as the proof of Theorem 2.38. \square

Theorem 2.44. Suppose $\vec{\alpha}, \vec{\beta}, \alpha, \beta \vdash \sigma(\vec{\alpha}, \vec{\beta}, \alpha, \beta)$ is a type as in Lemma 2.43. Then there exists a type $\text{rec } \alpha. \sigma(\vec{\alpha}, \vec{\beta}, \alpha, \alpha)$ with $\vec{\alpha}$ occurring only negatively and $\vec{\beta}$ only positively, and an isomorphism

$$i: \sigma(\vec{\alpha}, \vec{\beta}, \text{rec } \alpha. \sigma(\vec{\beta}, \vec{\alpha}, \alpha, \alpha), \text{rec } \alpha. \sigma(\vec{\alpha}, \vec{\beta}, \alpha, \alpha)) \multimap \text{rec } \alpha. \sigma(\vec{\alpha}, \vec{\beta}, \alpha, \alpha)$$

satisfying the conclusion of Lemma 2.44 with

$$\begin{aligned}
 \tau(\vec{\alpha}, \vec{\beta}) &= \text{rec } \alpha. \sigma(\vec{\alpha}, \vec{\beta}, \alpha, \alpha), \\
 \tau'(\vec{\alpha}, \vec{\beta}) &= \text{rec } \alpha. \sigma(\vec{\beta}, \vec{\alpha}, \alpha, \alpha), \\
 i &= \text{in}, \\
 \text{out} &= i^{-1}.
 \end{aligned}$$

Proof. Using Theorem 2.39, we can prove as in the proof of Lemma 2.40 that the pair

$$\begin{aligned}
 \text{out}^{-1}: \sigma(\vec{\alpha}, \vec{\beta}, \tau(\vec{\beta}, \vec{\alpha}), \tau'(\vec{\beta}, \vec{\alpha})) &\multimap \tau'(\vec{\beta}, \vec{\alpha}) \\
 \text{in}^{-1}: \tau(\vec{\beta}, \vec{\alpha}) &\multimap \sigma(\vec{\beta}, \vec{\alpha}, \tau'(\vec{\beta}, \vec{\alpha}), \tau(\vec{\beta}, \vec{\alpha}))
 \end{aligned}$$

also satisfies the conclusion of Lemma 2.44. Proceeding as in the proof of Lemma 2.41 we get an isomorphism $\tau(\vec{\alpha}, \vec{\beta}) \cong \tau'(\vec{\beta}, \vec{\alpha})$ up to internal equality, which implies the theorem. \square

The mixed induction/coinduction principle of Theorem 2.42 can be generalized to recursive types with parameters as follows.

Theorem 2.45. Suppose $\vec{R}_+: \text{AdmRel}(\vec{\omega}_+, \vec{\omega}'_+)$ and $\vec{R}_-: \text{AdmRel}(\vec{\omega}_-, \vec{\omega}'_-)$ are vectors of admissible relations, and

$$\begin{aligned}
 S_+ &: \text{AdmRel}(\text{rec } \alpha. \sigma(\vec{\omega}_-, \vec{\omega}_+, \alpha, \alpha), \text{rec } \alpha. \sigma(\vec{\omega}'_-, \vec{\omega}'_+, \alpha, \alpha)) \\
 S_- &: \text{Rel}(\text{rec } \alpha. \sigma(\vec{\omega}_+, \vec{\omega}_-, \alpha, \alpha), \text{rec } \alpha. \sigma(\vec{\omega}'_+, \vec{\omega}'_-, \alpha, \alpha))
 \end{aligned}$$

are relations. Then the following rule holds:

$$\frac{(i^{-1}, i^{-1}): S_- \multimap \sigma(\vec{R}_+, \vec{R}_-, S_+, S_-) \quad (i, i): \sigma(\vec{R}_-, \vec{R}_+, S_-, S_+) \multimap S_+}{S_- \subset \text{rec } \alpha. \sigma(\vec{R}_+, \vec{R}_-, \alpha, \alpha) \quad \text{rec } \alpha. \sigma(\vec{R}_-, \vec{R}_+, \alpha, \alpha) \subset S_+}$$

Proof. The proof proceeds as the proof of Theorem 2.42, and we start by considering the case where S_- is admissible. This time the terms generating h, h' have types

$$\begin{aligned} k &: \prod \vec{\alpha}, \vec{\beta}. \prod \omega', \omega. \\ &\quad (\sigma(\vec{\alpha}, \vec{\beta}, \omega', \omega) \multimap \omega) \multimap (\omega' \multimap \sigma(\vec{\beta}, \vec{\alpha}, \omega, \omega')) \multimap \text{rec } \alpha. \sigma(\vec{\alpha}, \vec{\beta}, \alpha, \alpha) \multimap \omega \\ k' &: \prod \vec{\alpha}, \vec{\beta}. \prod \omega, \omega'. \\ &\quad (\sigma(\vec{\alpha}, \vec{\beta}, \omega', \omega) \multimap \omega) \multimap (\omega' \multimap \sigma(\vec{\beta}, \vec{\alpha}, \omega, \omega')) \multimap \omega' \multimap \text{rec } \alpha. \sigma(\vec{\beta}, \vec{\alpha}, \alpha, \alpha) \end{aligned}$$

Now, notice first that

$$k \vec{\omega}_+ \vec{\omega}_- \text{rec } \alpha. \sigma(\vec{\omega}_+, \vec{\omega}_-, \alpha, \alpha) \text{rec } \alpha. \sigma(\vec{\omega}_-, \vec{\omega}_+, \alpha, \alpha) i i^{-1} = id_{\text{rec } \alpha. \sigma(\vec{\omega}_+, \vec{\omega}_-, \alpha, \alpha)} \quad (2.10)$$

$$k' \vec{\omega}_+ \vec{\omega}_- \text{rec } \alpha. \sigma(\vec{\omega}_+, \vec{\omega}_-, \alpha, \alpha) \text{rec } \alpha. \sigma(\vec{\omega}_-, \vec{\omega}_+, \alpha, \alpha) i i^{-1} = id_{\text{rec } \alpha. \sigma(\vec{\omega}_-, \vec{\omega}_+, \alpha, \alpha)} \quad (2.11)$$

$$k \vec{\omega}'_+ \vec{\omega}'_- \text{rec } \alpha. \sigma(\vec{\omega}'_+, \vec{\omega}'_-, \alpha, \alpha) \text{rec } \alpha. \sigma(\vec{\omega}'_-, \vec{\omega}'_+, \alpha, \alpha) i i^{-1} = id_{\text{rec } \alpha. \sigma(\vec{\omega}'_+, \vec{\omega}'_-, \alpha, \alpha)} \quad (2.12)$$

$$k' \vec{\omega}'_+ \vec{\omega}'_- \text{rec } \alpha. \sigma(\vec{\omega}'_+, \vec{\omega}'_-, \alpha, \alpha) \text{rec } \alpha. \sigma(\vec{\omega}'_-, \vec{\omega}'_+, \alpha, \alpha) i i^{-1} = id_{\text{rec } \alpha. \sigma(\vec{\omega}'_-, \vec{\omega}'_+, \alpha, \alpha)} \quad (2.13)$$

as in the proof of Theorem 2.42.

The theorem will follow from instantiating the parametricity schema of k, k' with \vec{R}_- substituted for $\vec{\alpha}$, \vec{R}_+ substituted for $\vec{\beta}$ and S_+ for ω and S_- for ω' . This tells us that if

$$\begin{aligned} (i^{-1}, i^{-1}) &: S_- \multimap \sigma(\vec{R}_+, \vec{R}_-, S_+, S_-) \\ (i, i) &: \sigma(\vec{R}_-, \vec{R}_+, S_-, S_+) \multimap S_+ \end{aligned}$$

then (using (2.10)-(2.13) above)

$$\begin{aligned} (id_{\text{rec } \alpha. \sigma(\vec{\omega}_+, \vec{\omega}_-, \alpha, \alpha)}, id_{\text{rec } \alpha. \sigma(\vec{\omega}'_+, \vec{\omega}'_-, \alpha, \alpha)}) &: S_- \multimap \text{rec } \alpha. \sigma(\vec{R}_+, \vec{R}_-, \alpha, \alpha) \\ (id_{\text{rec } \alpha. \sigma(\vec{\omega}_-, \vec{\omega}_+, \alpha, \alpha)}, id_{\text{rec } \alpha. \sigma(\vec{\omega}'_-, \vec{\omega}'_+, \alpha, \alpha)}) &: \text{rec } \alpha. \sigma(\vec{R}_-, \vec{R}_+, \alpha, \alpha) \multimap S_+ \end{aligned}$$

which was what we needed to prove.

For the general case, dropping the assumption that S_- is admissible, the proof proceeds exactly as in Theorem 2.42. \square

3. CONCLUSION

We have presented the logic LAPL for reasoning about parametricity in the domain theoretic case, and we have shown how in this logic Plotkin's encodings of recursive types can be verified. In later papers we will present a general notion of model of LAPL, and show how various earlier suggested domain theoretic models of parametric polymorphism fit this general notion of model. These models include a model based on admissible pers over a reflexive domain [BMP05], Rosolini and Simpson's construction in Synthetic Domain Theory [RS04] and a model based on the language Lily [BPR00].

In all these cases, a central point in verifying that these give rise to models of LAPL is to show that the various notions of admissible relations in the specific models satisfy the axioms for admissible relations presented in this paper. In the case of admissible pers the admissible relations are given by pointed chain complete subpers, in Synthetic Domain Theory the admissible relations are given by subdomain and in the case of Lily these are given by the $\top\top$ -closed relations. Of course the axioms presented here have been constructed to be general enough to fit all these cases.

As mentioned in the introduction, the logic LAPL can be seen as an axiomatization of a good category of domains. An interesting question is whether this is actually a useful

axiomatization, particularly because the solutions to recursive domain equations obtained using LAPL satisfy universal properties with respect to linear maps of PILL_Y , and most programming languages that one might want to model using domains do not correspond to linear calculi. A recent paper by the second author [Møg06] provides evidence of the usefulness of LAPL by showing how models of it give rise to models of FPC — a simply typed lambda calculus with general recursive types first suggested by Plotkin [Plo85] (see also [Fio96]) — and that these models model the expected reasoning principles for recursive types, reflecting a famous similar result in classical domain theory.

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